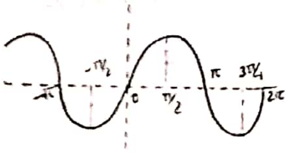
ExamsRoad.com

"Bringing Excellence to Students"

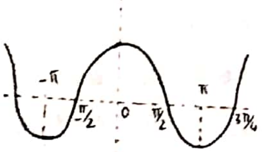
Trigonometry

Graphs:

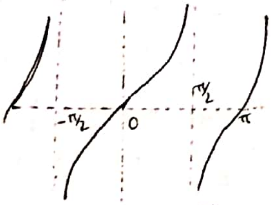
- $\sin x = y$



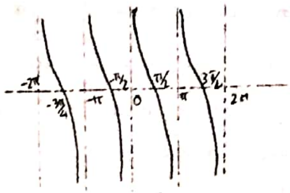
- $y = \cos x$



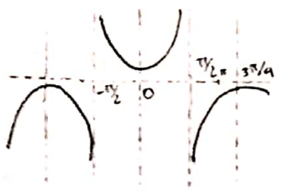
- $y = \tan x$



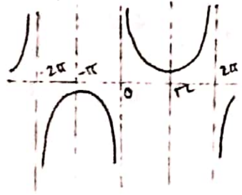
- $y = \cot x$



- $y = \sec x$



- $y = \operatorname{cosec} x$



Sum and Difference of two angles:

- $\sin(A+B) = \sin A\cos B + \cos A\sin B$
- $\sin(A-B) = \sin A\cos B - \cos A\sin B$
- $\cos(A+B) = \cos A\cos B - \sin A\sin B$
- $\cos(A-B) = \cos A\cos B + \sin A\sin B$
- $\tan(A\pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
- $\cot(A\pm B) = \dfrac{\cot A\cot B \mp 1}{\cot A + \cot B}$

Range of $f(\theta) = a\sin\theta + b\cos\theta$

$$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$$

Product into sum and difference

- $2\sin A\sin B = \sin(A+B) + \sin(A-B)$
- $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
- $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
- $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Imp Results:

- $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
- $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
- $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$
- $\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C - \sin A\sin B\cos C$
- $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

Multiple angles

- $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\overset{\sin}{\cancel{\cos}}\, 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
- $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$
- $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

Sum or difference into product.

- $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$
- $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Special Trios:

- $\cos A\cos(60-A)\cos(60+A) = \dfrac{\cos 3A}{4}$
- $\sin A\sin(60-A)\sin(60+A) = \dfrac{\sin 3A}{4}$
- $\tan A\tan(60-A)\tan(60+A) = \tan 3A$

AP of Angles:

- $\sin\alpha + \sin(\alpha+\beta) + \dots + \sin[\alpha+(n-1)\beta] = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\left[\sin\left(\alpha + \frac{(n-1)\beta}{2}\right)\right]$
- $\cos\alpha + \cos(\alpha+\beta) + \cos[\alpha+2\beta] + \dots + \cos[\alpha+(n-1)\beta] = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left[\alpha + \frac{(n-1)\beta}{2}\right]$

Conditional Identities $(A+B+C = \pi)$

- $\tan A + \tan B + \tan B = \tan A\tan B\tan C$
- $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$
- $\sin 2A + \sin 2B + \sin 2C = 14\sin A\sin B\sin C$
- $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

GP of Angles:

$$\cos A\cos 2A\cos 2^2A\cos 2^3A \cdots \cos 2^{n-1}A = \frac{\sin 2^n A}{2^n\sin A}$$

$A+B+C = \frac{\pi}{2}$

- $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A\sin B\sin C$
- $\cos^2 A + \cos^2 B + \cos^2 C = 2 + 2\sin A\sin B\sin C$

Inequalities: (ΔABC)

- $\tan A + \tan B + \tan C \ge 3\sqrt{3}$ [All acute]
- $\cos A + \cos B + \cos C \le \frac{3}{2}$
- $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$
- $\sec A + \sec B + \sec C \ge 6$ [Acute]
- $\operatorname{cosec}\frac{A}{2} + \operatorname{cosec}\frac{B}{2} + \operatorname{cosec}\frac{C}{2} \ge 6$ Eq

In ΔABC

- $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} \mp \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
- $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2}$
- $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \ge 1$

- $\tan 3A - \tan 2A - \tan A = \tan 3A\tan 2A\tan A$

Progression & Series

- Arithemic Progression- $a, a+d, a+2d, \cdots\cdots, a+(n-1)d$
- nth term (General term): $t_n = a+(n-1)d$, $t_n = l-(n-1)d$
- 3 terms in AP consideration- $a-d, a, a+d$
- 4 terms in A.P.- $a-3d, a-d, a+d, a+3d$
- If $a_1, a_2, a_3, \cdots\cdots a_{n-2}, a_{n-1}, a_n$ are in A.P., $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \cdots\cdots = a_r + a_{n-r+1}$
- Sum of n terms in an AP: $S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$
- If $S_n = an^2+bn+c$, $t_n = S_n - S_{n-1}$
- Arithmetic mean: $AM(x_1, x_2) = \frac{x_1+x_2}{2}$, $AM(x_1, x_2, x_3, \cdots, x_n) = \frac{x_1+x_2+x_3+\cdots+x_n}{n}$
- $A_1 + A_2 + \cdots + A_n = n\left(\frac{a+b}{2}\right)$

- Geometric Progression- $a, ar, ar^2, \cdots, a^{n-1}$
- nth term (General term): $t_n = ar^{n-1}$
- Sum of n terms: $S_n = \frac{a(1-r^n)}{1-r}$ $[1>r]$ $= \frac{a(r^n-1)}{r-1}$ $[r>1]$
- If $x_1, x_2, x_3 \cdots$ are in GP, $\log x_1, \log x_2 \cdots$ are in AP
- 3 terms in GP: $\frac{a}{r}, a, ar$
- 4 terms in GP: $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$
- Sum of Infinite GP. $S_\infty = \frac{a}{1-r}$ $[|r|<1]$
- Geometric means: $GM(x_1, x_2) = (x_1 x_2)^{1/2}$, $GM(x_1, x_2, x_3, \cdots, x_n) = (x_1 x_2 x_3 \cdots x_n)^{1/n}$
- $G_1 G_2 G_3 \cdots G_n = (\sqrt{ab})^n$

- Harmonic progression: $a_1, a_2, a_3 \cdots$ are in H.P. if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \cdots$ are in AP.
- nth term (General term): $t_n = \frac{1}{t_n \text{ of AP}}$
- Harmonic Mean: $HM(x_1, x_2) = \frac{2x_1x_2}{x_1+x_2}$, $HM(x_1, x_2 \cdots x_n) = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}$

- Inequalities: $A \geq G \geq H$
- $G^2 = AH$

- Special series: $\Sigma n = \frac{n(n+1)}{2}$, $\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$, $\Sigma n^3 = \frac{n^2(n+1)^2}{4}$

- Arithmetico Geometric Progression (AGP): $S = a + (a+d)r + (a+2d)r^2 + \cdots$

$$\begin{aligned} S &= a + (a+d)r + (a+2d)r^2 + \cdots \\ -\ rS &= \quad ar + (a+d)r^2 + \cdots \\ \hline S(1-r) &= a + d(r + r^2 + \cdots) \end{aligned}$$

- Difference Series:

$$\begin{aligned} S &= 1 \overset{1}{+} 2 \overset{2}{+} 4 \overset{3}{+} 7 \overset{4}{+} 11 \overset{5}{+} 16 + \cdots + t_n \\ -\ S &= \quad 1 + 2 + 4 + 7 + 11 + \cdots + t_n \\ \hline 0 &= 1 + (1+2+3+\cdots) - t_n \end{aligned}$$

- $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)} = \sum_{r=1}^{n} \frac{1}{3}\left[\frac{(r+3)-r}{r(r+1)(r+2)(r+3)}\right]$
- $\sum_{r=1}^{n} r(r+1)(r+2)(r+3) = \sum_{r=1}^{n} \frac{1}{5}\left((r+4)-(r-1)\right)\left(r(r+1)(r+2)(r+3)\right)$

Weighted mean:

- $\frac{a_1^m + a_2^m + \cdots + a_n^m}{n} > \left(\frac{a_1+a_2+a_3+\cdots+a_n}{n}\right)^m$ If $m<0$ or $m>1$
- $\frac{a_1^m + a_2^m + \cdots + a_n^m}{n} < \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m$ If $0<m<1$

Permutation & Combination

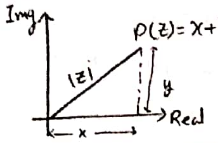
- Let p be a given prime and n, any positive integer, Then the maximum power of p present in $n!$ is $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots$ Where $[\cdot]$ → Greatest Integer function
- ${}^nC_r \times r! = {}^nP_r$
- Number of permutations of n different things taken r at a time → ${}^nP_r = \frac{n!}{(n-r)!}$
- Number of permutations of n different things taken all at a time → $n!$
- Number of permutation of n things [p are allike, q are alike, r are alike] → $\frac{n!}{p!\,q!\,r!}$
- Number of combinations (selections) of n different things taking r at a time → ${}^nC_r = \frac{n!}{r!(n-r)!}$
- ${}^nC_r = {}^nC_{n-r}$, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, $r \cdot {}^nC_r = n\,{}^{n-1}C_{r-1}$
- $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$
- $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$
- When n is even, max value of ${}^nC_r \longrightarrow {}^nC_{n/2}$
  When n is odd, max value of ${}^nC_r \longrightarrow {}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$
- No of ways of arranging n different things in circular manner → $(n-1)!$
- When ACW/CW doesn't matter (e.g - necklace, garland), circular arrangement → $\frac{(n-1)!}{2}$
- Total no of combination of n things taken 1 or more at a time → ${}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n = 2^n - 1$
- Total no of selections of n things, [p similar, q similar, r alike] (including 0) → $(p+1)(q+1)(r+1)$
- If $N = p_1^a \times p_2^b \times p_3^c \times \cdots$ where $a, b, c, \ldots$ are ~~are~~ non~~ga~~negative integers, $p_1, p_2, p_3 \ldots$ are prime no.
  Then → Total No of Divisors $= (a+1)(b+1)(c+1)\cdots$
  - Sum of all divisors $= \left(\frac{p_1^{a+1}-1}{p_1-1}\right) \times \left(\frac{p_2^{b+1}-1}{p_2-1}\right) \times \left(\frac{p_3^{c+1}-1}{p_3-1}\right)\cdots$
  - All the divisors excluding 1 and N are called proper divisors
  - No of ways of writing N as a product of two natural nos → $\begin{cases} \left[\frac{1}{2}(a+1)(b+1)(c+1)\cdots\right] & \text{if } N \text{ isn't a perfect square} \\ \left[\frac{1}{2}(a+1)(b+1)(c+1)\cdots+1\right] & \text{if } N \text{ is a perfect square} \end{cases}$
  - N is a perfect square if $a, b, c \ldots$ all are even
  - N is a perfect cube if $a, b, c \ldots$ all are multiples of 3.
  - $N = 2^a \times 3^b \times 5^c \times \cdots$
    - If N is odd, $a = 0$, $b, c, d \cdots \geq 0$
    - If N is even, $a \geq 1$, $b, c, \ldots \geq 0$
- No of Non negative integral sol^n of the eq^n $x_1 + x_2 + x_3 + \cdots + x_r = n$ is → ${}^{n+r-1}C_{r-1}$
- No of positive integral sol^n of the eq^n $x_1 + x_2 + x_3 + \cdots + x_r = n$ is → ${}^{n-1}C_{r-1}$
- Sum of all n-digit numbers formed using n digits $= (n-1)!\,(\text{Sum of all } n \text{ digits}) \times (\underbrace{111\ldots\ldots1}_{n \text{ times}})$
- No of diagonals of n sided polygon → ${}^nC_2 - n = \frac{n(n-3)}{2}$
- No of squares ~~of~~ in two system of perpendicular parallel lines (When 1st set contains m lines and 2nd set contains n lines) is equal to → $\sum_{r=1}^{m-1}(m-r)(n-r)$; $(m<n)$
- Dearrangements : No of ways so that no letter goes to the correct address.

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots\cdots + (-1)^n \frac{1}{n!}\right]$$

Complex Number

- $z = x + iy$, $x, y \in R$ and $i = \sqrt{-1}$ ▨ $Re(z) = x$, $Im(z) = y$ ▨ $\sqrt{-a} = i\sqrt{a}$
- The property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is valid only if at least one of a and b is non negative, If a and b are both negative, then $\sqrt{a}\sqrt{b} = -\sqrt{|a||b|}$
- If $a+ib = c+id$, $a = c$, $b = d$
- $a+ib > c+id$ is meaning ful only if $b = d = 0$
- In real no system, $a^2+b^2=0$, $a=b=0$. But $z_1^2+z_2^2=0$ does NOT mean $z_1 = z_2 = 0$.
- $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$; $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, $i^{4n} = 1$
- **Square Root of a Complex No:** $\sqrt{a+ib} = x+iy \Rightarrow a = x^2-y^2$; $2xy = b$ solve.
  Sign of b decides whether x and y are of same sign or opposite sign.

Modulus of CN: $|z| = r = \sqrt{x^2+y^2}$



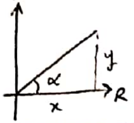
Amplitude of CN

Argument / amplitude of CN , $\theta = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{Im(z)}{Re(z)}\right)$

↳ From the real axis . $\arg(z) \in [-\pi, \pi]$

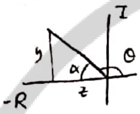
Principle Argument

▨ Quad I / $(x>0, y>0)$



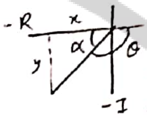
$\arg(z) = \alpha$
$= \tan^{-1}|y/x|$

▨ Qud II / $(x<0, y>0)$



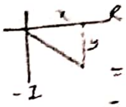
$\arg(z) = \theta$
$= \pi - \alpha$
$= \pi - \tan^{-1}|y/x|$

▨ Qud III / $(x<0, y<0)$



$\arg(z) = \theta$
$= \alpha - \pi$
$= \tan^{-1}\left|\frac{y}{x}\right| - \pi$

▨ Qud IV /



$\arg(z) = \theta$
$= -\alpha$
$= -\tan^{-1}\left|\frac{y}{x}\right|$

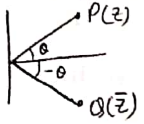
Polar form

$z = x+iy = r(\cos\theta + i\sin\theta)$

Euler's form $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$

- Conjugate of a CN:



$z = x+iy$

$\bar{z} = x-iy$

$z + \bar{z} = 2Re(z)$

$z - \bar{z} = 2Im(z)$

Properties of Conjugates

\# $(\bar{\bar{z}}) = z$ \# if $z = \bar{z}$, z is purely real \# $\overline{z_1z_2} = \bar{z_1}\cdot\bar{z_2}$

\# $z+\bar{z} = 0$, z is purely imaginary \# $\overline{z^n} = (\bar{z})^n$ \# $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$

\# $z\cdot\bar{z} = z^2$ \# $\overline{z_1+z_2} = \bar{z_1}+\bar{z_2}$

Properties of modulus

\# $|z| = 0 \Rightarrow z = 0 = Im(z) = Re(z)$

\# $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

\# $-|z| \leq Re|z| \leq |z|$

\# $-|z| \leq Im|z| \leq |z|$

\# $z\cdot\bar{z} = |z|^2$ \# $|z^n| = |z|^n$

\# $|z_1z_2\cdots z_n| = |z_1||z_2|\cdots|z_n|$

\# $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

\# $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2Re(z_1\bar{z_2})$

\# $|z_1-z_2|^2 = |z_1|^2+|z_2|^2-2Re(z_1\bar{z_2})$

\# $|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$

\# $|z_1-z_2| \rightarrow$ dist b/w z_1 & z_2

\# $|z_1+z_2| \leq |z_1|+|z_2|$

\# $|z_1+z_2+\cdots+z_n| \leq |z_1|+|z_2|+\cdots+|z_n|$

\# $|z_1+z_2| \geq ||z_1|-|z_2||$

Properties of Arguments

\# $\arg(z_1z_2) = \arg(z_1)+\arg(z_2)$

\# $\arg(z_1z_2\cdots z_n) = \arg(z_1)+\cdots+\arg(z_n)$

\# $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

\# $\arg(\bar{z}) = -\arg(z)$

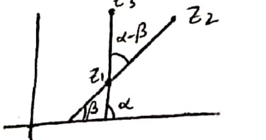
\# $\arg(z^n) = n\arg(z)$

\# $\arg\left(\frac{1}{z}\right) = -\arg(z)$

\# If z is purely imaginary,
$\arg(z) = \pm\pi/2$

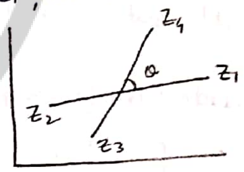
\# If z is purely real,
$\arg(z) = 0/\pi$

- Angle b/w 2 lines



$\alpha - \beta = \arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$

- Angle b/w line joining z_1, ~~and~~ z_2 & z_3, z_4



$\theta = \arg\left(\frac{z_4-z_3}{z_1-z_2}\right)$

- If z_1 , z_2 and z_3 are vertices of an equilateral triangle. Then

$$\frac{1}{z_1-z_2}+\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1} = 0$$

- i.e.

$$z_1^2+z_2^2+z_3^2 = z_1z_2+z_2z_3+z_3z_1$$

- Square Root of $z = a+ib$ are

$$\begin{cases} \pm\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}} \text{ , for } b>0 \\ \pm\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}} \text{ , for } b<0 \end{cases}$$

- If z_1, z_2, z_3 are vertices of an isosceles right angled triangle, w/ right angle at z_3, The

$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = (re^{i\theta})^n = r^n(e^{in\theta}) = (\cos(n\theta) + i\sin(n\theta))$

- $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$
- $\frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$
- $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos\theta$
- $(\cos\theta_1 - i\sin\theta_2)^n \neq \cos n\theta_1 - i\sin n\theta_2$
- $(\sin\theta + i\cos\theta)^n = [\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)]^n = [\cos n(\pi/2 - \theta) + i\sin n(\pi/2 - \theta)]$

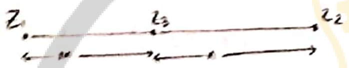
Cube Roots of Unity

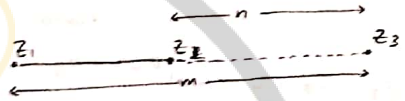
$z = 1^{1/3} = 1, \omega, \omega^2$ where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\omega = e^{i2\pi/3}$, $\omega^2 = e^{-i2\pi/3}$

- Sum of roots is 0; $1 + \omega + \omega^2 = 0$
- Product of roots $= 1$; $1 \cdot \omega \cdot \omega^2 = 1$
- $\omega = \frac{1}{\omega^2}$, $\omega^2 = \frac{1}{\omega}$
- $\omega = \overline{\omega^2}$, $\omega^2 = \overline{\omega}$
- $\omega^{3n+1} = \omega$, $\omega^{3n+2} = \omega^2$, $\omega^{3n} = 1$
- $1 + \omega^n + \omega^{2n} = \begin{cases} 3, & n \text{ is a multiple of } 3 \\ 0, & n \text{ is not a multiple of } 3 \end{cases}$
- Cube roots of unity represent the vertices of an equilatoral triangle on Argand Plane

Section Formula





Internally, $z_3 = \frac{mz_2 + nz_1}{m + n}$

Externally, $z_3 = \frac{mz_2 - nz_1}{m - n}$

- Centroid of Δ formed by z_1, z_2 and $z_3 \rightarrow \frac{z_1 + z_2 + z_3}{3}$
- If circumcentre of an Δ is origin, then orthocentre $\rightarrow z_1 + z_2 + z_3$

nth root of unity

- Sum of all nth roots of unity $= 0$
- Product of all roots $= 1 \cdot \alpha \cdot \alpha^2 \cdot \alpha^3 \cdots \alpha^{n-1} = \begin{Bmatrix} 1, & n \text{ is odd} \\ -1, & n \text{ is even} \end{Bmatrix}$

$z = 1^{1/n} \Rightarrow z = (\cos 2k\pi + i\sin 2k\pi)^n = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k$

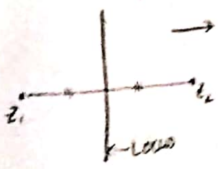
$\therefore \alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

Locus of a CN

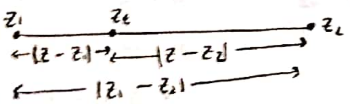
(z_1 and z_2 are fixed, z is a variable point)

$\rightarrow |z - z_1| = |z - z_2|$

$\rightarrow$ z lies on perpendicular bisector of z_1z_2



$\rightarrow |z - z_1| + |z - z_2| = |z_1 - z_2|$

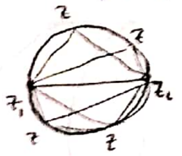


$\rightarrow$ z lies on the segment joining z_1 and z_2

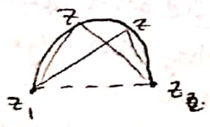
$\rightarrow |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

$\rightarrow$ Circle with z_1 and z_2 as diameter extremeties.

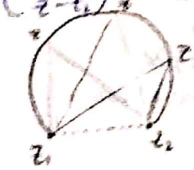
$\rightarrow \arg\left(\frac{z - z_1}{z - z_2}\right) = \pm \pi/2$



$\rightarrow \arg\left(\frac{z - z_1}{z - z_2}\right) = \pi/2$

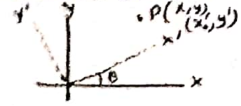


$\rightarrow \arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ (fixed)



Straight Line

Rotation of Axes



$x = x'\cos\theta - y\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

	$x\downarrow$	$y\downarrow$
$x'\rightarrow$	$\cos\theta$	$\sin\theta$
$y'\rightarrow$	$-\sin\theta$	$\cos\theta$

Distance formula.

$$|d| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

- If A, B, C are collinear, $[\Delta ABC] = 0$
- If A, B, C have rational coordinates, ΔABC isn't equilateral.

Area of triangle

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Stain Method

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} = \frac{1}{2}\left[(x_1y_2+x_2y_3+x_3y_1)-(x_2y_1+(x_3y_2)+x_1y_3)\right]$$

$$= \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

Area of polygon

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_y \end{vmatrix}$$

Points must be taken in cyclic order

Section Formula

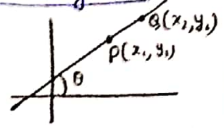
- Internal: $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$
- External: $\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$

Special points:

- Centroid: $\left(\frac{\Sigma x}{3}, \frac{\Sigma y}{3}\right)$
- Incentre: $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$
- Circumcentre: $\left(\frac{x_1\sin 2A+x_2\sin 2B+x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \text{similar}\right)$
- O, G, H of an acute angle triangle are collinear

$O \overset{1}{—} G \overset{2}{—} H$ (1 : 2)

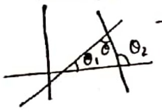
Straight line



Eqⁿ of line → $\boxed{y = mx + c}$

$$\boxed{\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}}$$

Angle b/w two lines



$$\tan\theta = \pm\frac{m_1-m_2}{1+m_1m_2}$$

- If $m_1 = m_2$ → lines are parallel
- If $m_1m_2 = -1$ → lines are perpendicular.

Eqⁿ of line

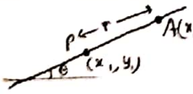
- Parallel to x axis → $y = b$
- Parallel to y axis → $x = a$
- slope Intercept form → $y = mx + c$
- Point slope form → $y - y_1 = m(x - x_1)$
- Two point form → $\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$
- Intercept form → $\frac{x}{a} + \frac{y}{b} = 1$
- Normal form → $x\cos\alpha + y\sin\alpha = p$

Conditions

$a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

- Coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- Intersecting if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$
- Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
- Perpendicular if $a_1a_2 + b_1b_2 = 0$

Parametric form



$A(x,y) = (x_1 + r\cos\theta, y_1 + r\sin\theta)$

$ax + by + c = 0$ →

- ∥ to line → $ax + by + \lambda = 0$
- ⊥ to line → $bx - ay + \mu = 0$

Dist of a point from a line

$$|d| = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Dist b/w two parallel lines

$$d = \frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$$

Concurrency of three lines

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Angle bisector

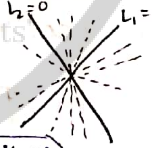
of $a_1x + b_1y + c_1 = 0$ & $a_1x + b_2y + c_2 = 0$ →

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \rightarrow ⓧ \quad / \quad \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \rightarrow ⓨ$$

- Make c_1, c_2 +ve. Then (x) contains origin

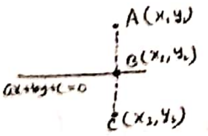
	Acute bisec	Obtuse bisec	
$a_1a_2+b_1b_2>0$	(Y)	(x)	Origin is in obtuse
$a_1a_2+b_1b_2<0$	(x)	(Y)	Origin is in acute.

Family of St. lines



$$L_1 + \lambda L_2 = 0$$

For Foot of Perpendicular / Image



B// $\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$

C// $\frac{x_3-x_1}{a} = \frac{y_3-y_1}{b} = -\frac{2(ax_1+by_1+c)}{a^2+b^2}$

Pair of St. lines

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$$

Bisector of Angle b/w pair of st. line

$ax^2 + 2hxy + by^2 = 0$

↳ $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

Angle b/w pair of St. lines

$m_1 + m_2 = -\frac{2h}{b}$; $m_1m_2 = \frac{a}{b}$ $\quad \therefore \tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$

- $a+b=0$ → lines are perpendicular
- $h^2=ab$ → lines are ∥ or ~~coind~~ coincident.

General 2nd degree equation

$ax^2 + 2hxy + bx^2 + 2gx + 2hy + c = 0$ →

- $\Delta = 0$ (Pair of st. line
- $\Delta \neq 0, a = b, h = 0$ (Circle)
- $\Delta \neq 0, h^2 = ab$ (Parabola)
- Ellipse ($\Delta \neq 0, h^2 < ab$)
- $\Delta \neq 0, h^2 > ab$ (Hyperbola)

Pair of St. line if

$\Delta(abc + 2fgh - af^2 - bg^2 - ch^2) = 0$

- Point of intersection → $\left(\frac{bg-hf}{h^2-ab}, \frac{af-gf}{h^2-ab}\right)$

Circle

Eqn of Circle

$(x-h)^2+(y-k)^2=r^2$

- If centre $(0,0) \rightarrow x^2+y^2=r^2$

- From general equation

$x^2+y^2+2gx+2fy+c=0$

- Centre $(-g,-f)$
- radius $=\sqrt{g^2+f^2-c}$

- From diametre extremeties:

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

- Intercepts made on axis

$x^2+y^2+2gx+2fy+c=0$

x int $\rightarrow 2\sqrt{g^2-c}$

y int $\rightarrow 2\sqrt{f^2-c}$

- Equation of circumcircle of Δ formed by $ax+by+c=0$ w/ coordinate axis.

$ab(x^2+y^2)+c(bx+ay)=0$

Tangents

$x^2+y^2+2gx+2fy+c=0$

$\xrightarrow{T}$ $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

- For $x^2+y^2=a^2$:

Point form $\rightarrow xx_1+yy_1-a^2=0$

Parametric $\rightarrow x\cos\theta+y\sin\theta-a=0$

Slope form $\rightarrow y=mx\pm a\sqrt{1+m^2}$

- For $x^2+y^2+2gx+2fy+c=0$

$y+f=m(x+g)\pm\sqrt{g^2+f^2-c}\sqrt{1+m^2}$

- From a point outside

$(y-y_1)=m(x-x_1)$

- Length of tangent from a point to a circle

$d=\sqrt{S_1}$

- Pair of tangents (combined eqn)

$SS_1=T^2$

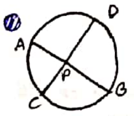
- Two lines ~~cut~~ $a_1x+b_1y+c_1=0$; $a_2x+b_2y+c_2=0$ cut the coordinate axes at concyclic points. if $m_1m_2=1 \rightarrow a_1a_2=b_1b_2$

Parametric Form

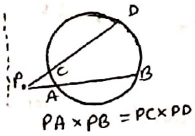
$x^2+y^2=r^2 \rightarrow (r\cos\theta, r\sin\theta)$ $[0\leq\theta<2\pi]$

$(x-h)^2+(y-k)^2=r^2 \longrightarrow (h+r\cos\theta, k+r\sin\theta)$

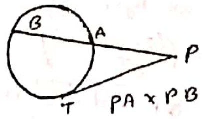
- A line intersects, touches or doesnt intersect the circle if radius is greater than, equal to or less than the length of perpendicular from centre of the circle to the line.



$PA\times PB=PC\times PD$



$PA\times PB=PC\times PD$



$PA\times PB=PT^2$

- If two lines $a_1x+b_1y+c=0$ and $a_2x+b_2y+c_2=0$ meet the axes in four distinct ~~points~~ concyclic points, then $a_1a_2=b_1b_2$ and also the eqn of the circle passing thru those concyclic points is.

$(a_1x+b_1y+c_1)(a_2x+b_2y+c_1)-(a_1b_2+a_2b_1)xy=0$

~~Tangents~~

Normals

$$(y-y_1)=\frac{y_1+f}{x_1+g}(x-x_1)$$

- Chord of Contact $\longrightarrow T=0$
- Eqn of chord bisected at $(x_1,y_1) \longrightarrow S_1=T$

Director circle

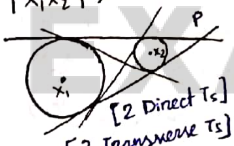
$x^2+y^2=a^2 \xrightarrow{DC} x^2+y^2=2a^2$ $[r_{DC}=\sqrt{2}\cdot r]$

- Angle of Intersection of two circles: $\longrightarrow \cos\theta=\left|\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right|$

  - When $\theta=90°$ [Orthogonally] $\longrightarrow 2g_1g_2+2f_1f_2=c_1+c_2$

Intersections

- $|x_1x_2|>r_1+r_2$



[2 Direct Ts]

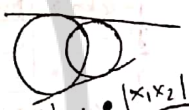
[2 Transverse Ts]

[P divided x_1x_2 externally in ratio $r_1:r_2$]

- $|x_1x_2|=r_1+r_2$

[Total 3 Ts]

- $|r_1-r_2|<|x_1x_2|<|r_1+r_2|$

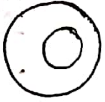


[Total 2 Common Ts]

- $|x_1x_2|=|r_1-r_2|$

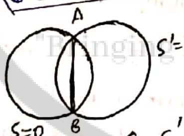
[1 CTs]

- $|x_1x_2|<|r_1-r_2|$



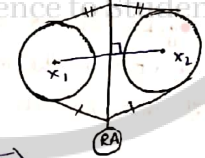
[No CT]

Common Chord



$AB \rightarrow S-S'=0$

Radical Axis



- RA is ⊥ to line joining the centres
- RA bisects common tangents
- Need not pass thru mp of x_1x_2
- If 2 circles cut a third circle orthogonally, RA of these 2 pass thru 3rd one's centre.

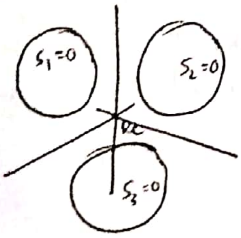
Length of common tangents

DCT $\rightarrow |AB|=\sqrt{d^2-(r_1-r_2)^2}$

TCT $\rightarrow |CD|=\sqrt{d^2-(r_1+r_2)^2}$

[d → dist b/w two centres]

Radical Centre



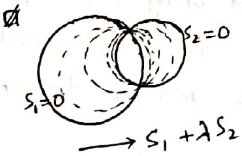
Solve,

$S_1-S_2=0$

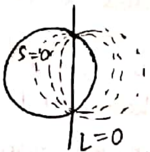
$S_2-S_3=0$

$S_3-S_1=0$

Family of Circles



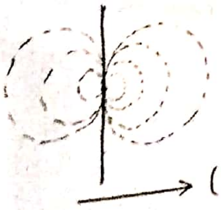
$\longrightarrow S_1+\lambda S_2=0$



$\longrightarrow S+\lambda L=0$

$\longrightarrow S+\lambda L=0$

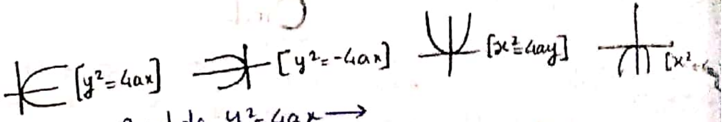
- Circles touch a line at (x_1,y_1):



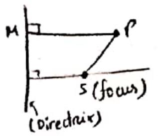
$\longrightarrow (x-x_1)^2+(y-y_1)^2+\lambda L=0$

PARABOLA

- Standard forms →



Eccentricity

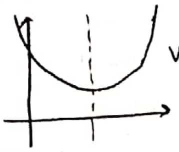


$\frac{PS}{PM} = e$

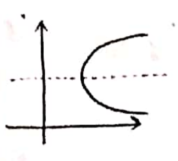
$e = 0 \rightarrow$ circle
$e = 1 \rightarrow$ Parabola
$e < 1 \rightarrow$ Ellipse
$e > 1 \rightarrow$ Hyperbola
$e = \infty \rightarrow$ Pair of st. lines

- Position of a point w.r.t. a Parabola $y^2 = 4ax \rightarrow$ $y^2 - 4ax > 0 \rightarrow$ outside / $y^2 - 4ax = 0 \rightarrow$ on / $y^2 - 4ax < 0 \rightarrow$ Inside

- Parabolic curve



$y = Ax^2 + Bx + C$
Vertex $\rightarrow \left(-\frac{B}{2A}, -\frac{D}{2A}\right)$
$LR \rightarrow \frac{1}{|A|}$



$x = Ay^2 + By + C$
Vertex $\rightarrow \left(-\frac{D}{4A}, -\frac{B}{2A}\right)$
Length $LR \rightarrow \frac{1}{|A|}$

- Equation of Conic:

Focus (α, β), Directrix $(ax + by + c = 0)$

$\rightarrow (x-\alpha)^2 + (y-\beta)^2 = e^2\left(\frac{(ax+by+c)^2}{a^2+b^2}\right)$

Parametric form

$(y-k)^2 = 4a(x-h)$

$\rightarrow x = h + at^2$
$y = k + 2at$

Equation of tangents

- Point form →

$yy_1 = 2a(x + x_1)$

- Parametric form →

$ty = x + at^2$ [at $(at^2, 2at)$]

- Slope form →

$y = mx + a/m$ [at $(a/m^2, 2a/m)$]

Equation of Normal

- Point form →

$y - y_1 = -\frac{y_1}{2a}(x - x_1)$

- Parametric form →

$y = -tx + 2at + at^3$

- slope form →

$y = mx - 2am - am^3$

Properties of Focal Chord

If $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is a focal chord, then $t_1t_2 = -1$, $Q \equiv \left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$

- Chord joining $P(at_1^2, 2at_1)$ has length $a(t + 1/t)^2$
- Focal chord from $P(at^2, 2at)$ has length $4a\cosec^2\theta$
- Focal chord making angle θ with axis has length
- Semi latus Rectum is HM of SP & SQ whe P and Q are extremities of focal chord, $S \rightarrow$ focus
- Circle described on the focal length as diameter touches tangent at vertex
- Circle described on the focal chord as diameter touches directrix.

- Pair of tangents: $\rightarrow SS_1 = T^2$
- Chord of contact $\rightarrow T = 0$
- Equation of chord from mp. $\longrightarrow S_1 = T$

Properties of Tangents

Tangents at two points $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ on the Parabola $y^2 = 4ax$

- Point of Intersection of tangents at two points is $T(at_1t_2, a(t_1+t_2))$
- Locus of foot of ⊥ from focus upon any tangent is tangent at vertex.
- Length of Tangent b/w POC and POI w/ directrix subtends 90° at the focus.
- Tangents at Extremeties of focal chord are perpendicular ~~are~~ on Directrixs.

Parabola	Normal (Parametric)	Normal (Slope)
$y^2 = 4ax$	$y = -tx + 2at + at^3$ $(at^2, 2at)$	$y = mx - 2am - am^3$ $(am^2, -2am)$
$y^2 = -4ax$	$y = tx + 2at + at^3$ $(-at^2, 2at)$	$y = mx + 2am + am^3$ $(-am^2, 2am)$
$x^2 = 4ay$	$x = -ty + 2at + at^3$ $(2at, at^2)$	$y = mx + 2a + a/m^2$ $\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$
$x^2 = -4ay$	$x = ty + 2at + at^3$ $(2at, -at^2)$	$y = mx - 2a - a/m^2$ $\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$

Properties of Normals

- Normals other than axis of Parabola never passes thru focus.
- POI of Normals from $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ $\rightarrow [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$
- Normal at point $P(t_1)$ meets the curve again at $Q(t_2)$, $t_2 = -t_1 - 2/t_1$

Co-normal Points

$y = mx - 2am - am^3 \Rightarrow am^3 + (2a - h)m + k = 0$ (cubic in m)

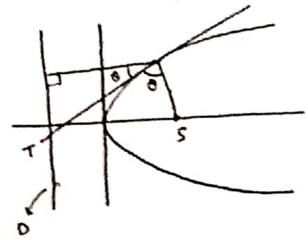
$m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$

- Algebraic sum of ordinates of co-normal points $= 0$
- Centroid of the triangle formed by them lies on axis.
- If three normals drawn on $y^2 = 4ax$ from (h, k) is real $\rightarrow \boxed{h > 2a}$

Reflection Property of Parabola

The tangent at any point P to a parabola bisects the angle between the focal chord through P and perpendicular from P to the directrix

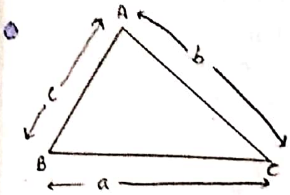
- Thus, if any light ray is sent along a line parallel to the axis of the parabola then the reflected ray passes thru the focus, as the normal bisects the angle between the incident ray and reflected ray.



- Tangents are drawn from the point (x_1, y_1) to the parabola $y^2 = 4ax$, the length of the chord of contact $= \frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$

- Area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$

Properties of Triangle / Solution of Triangle



$a+b+c = 2s$ (Perimeter)

$s = \frac{a+b+c}{2}$ (semi perimeter)

Sine Rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = 2R$$

[R → circumradius]

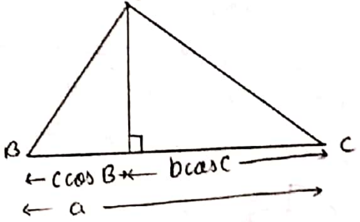
Cosine Rule

$\# \cos A = \frac{b^2+c^2-a^2}{2bc}$

$\# \cos B = \frac{c^2+a^2-b^2}{2ca}$

$\# \cos C = \frac{a^2+b^2-c^2}{2ab}$

Projection Formula



$\# a = b\cos C + c\cos B$

$\# b = c\cos A + a\cos C$

$\# c = a\cos B + \cancel{B\cos b}\ b\cos A$

Napier Formula

$\# \tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

$\# \tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$

$\# \tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$

Half Angle formula

$\# \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$\# \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}}$

$\# \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

$\# \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

$\# \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$

$\# \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

$\# \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$\# \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$\# \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Area of Triangle

$\# \Delta = \frac{1}{2}\cdot b\cdot h$

$\# \Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$

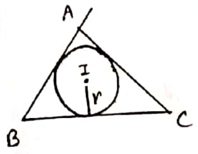
$\# \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\# \Delta = r\times s$ (r is Inradius)

$\# \Delta = 2R^2\sin A\sin B\sin C$

$\# \Delta = \frac{abc}{4R}$ (R is circumcentre)

Inradius

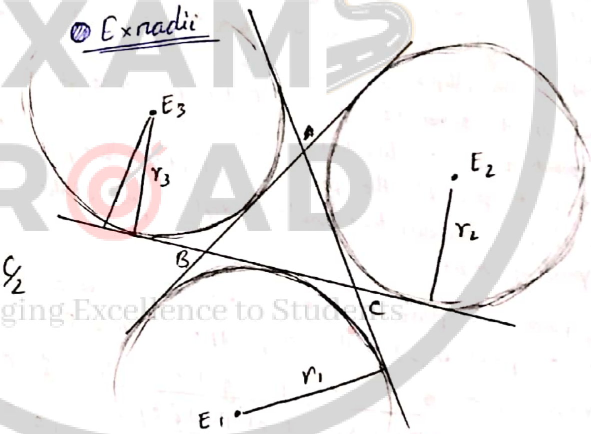


$\# r = \frac{\Delta}{s}$

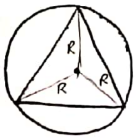
$\# r = (s-a)\tan\frac{A}{2}$
$= (s-b)\tan\frac{B}{2}$
$= (s-c)\tan\frac{C}{2}$

$\# r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Exradii



Circumradius



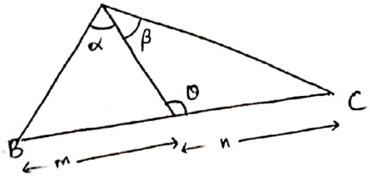
$\# R = \frac{abc}{4\Delta}$

$\# r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$\# r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$\# r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

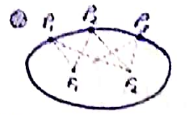
m−n cot Theorem



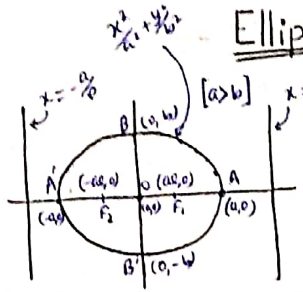
$\# (m+n)\cot\theta = m\cot\alpha - n\cot\beta$

$\# (m+n)\cot\theta = n\cot\beta - m\cot C$

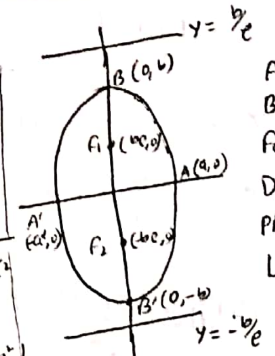
Ellipse



$P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2$
$= P_3F_1 + P_3F_2$



AA'(major axis) = 2a
BB'(minor axis) = 2b
foci = $(\pm ae, 0)$
Directrix → $x = \pm \frac{a}{e}$
$PF_1 + PF_2 = 2a$
$b^2 = a^2(1-e^2) / e = \sqrt{1-\frac{b^2}{a^2}}$
Vertices = $(\pm a, 0)$
Latus Rectum = $\frac{2b^2}{a}$
End of LR = $(\pm ae, \pm \frac{b^2}{a})$



[b>a]
AA'(minor axis) = 2a
BB'(major axis) = 2b
Foci → $(0, \pm be)$
Directrix → $y = \pm b/e$
$Pf_1 + Pf_2 = 2b$
$LR = \frac{2a^2}{b}$
Ends of LR = $(\pm \frac{a^2}{b}, \pm b$

- Two ellipses are similar if they have equal eccentricity.
- Ellipse with axes ∥ to coordinate axes and centre $(h,k) \rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
- Length of LR = $(\text{minor axis})^2/\text{major axis} = 2e\left(\frac{a}{e} - ae\right)$
- Eqⁿ of an Ellipse referred to two perpendicular lines.

$L_1 : a_1x + b_1y + c_1 = 0$
$L_2 : b_1x - a_1y + c_2 = 0$

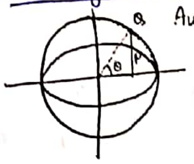
$$\rightarrow \frac{\left(\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}\right)^2}{a^2} + \frac{\left(\frac{b_1x-a_1y+c_2}{\sqrt{a_1^2+b_1^2}}\right)^2}{b^2} = 1$$

  - Centre at intersection point of L_1 & L_2
  - Major axis is along L_2 $(a > b)$

- Position of a point w.r.t an Ellipse (h, k)

$\frac{x^2}{a^2} + \frac{k^2}{b^2} - 1 >, =, < 0$ → outside, on, inside.

- Auxiliary Circle / Eccentric Angle



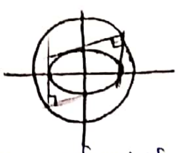
Aux circle: $x^2+y^2=a^2$
Q $(a\cos\theta, a\sin\theta)$
P $(a\cos\theta, b\sin\theta)$
θ is called eccentric angle of point P

- Pair of tangents
$SS_1 = T^2$
- COC
$T = 0$
- Eqⁿ of chord with mp (x_1, y_1)
$T = S_1$

- Properties of ~~auxiliary circle~~ ellipse.
  - Area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.
  - Ratio of area of any triangle PQR inscribed in ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and that of triangle formed by corresponding points on the aux circle is b/a.
  - Semi LR is HM of segments of focal chord.
  - Circle describe on focal length as diameter always touches auxiliary circle.

- Equation of tangent
  - Point form → $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
  - Parametric form → $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$
    $(a\cos\theta, b\sin\theta)$
  - Slope form → $y = mx \pm \sqrt{a^2m^2+b^2}$

- Director circle



locus of poi of ⊥ tangents.

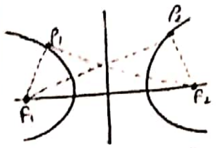
- Important Properties related to tangents
  - Locus of feet of perpendiculars from foci upon any ~~one~~ tangent is an auxiliary circle.
  - Product of lengths of perpendiculars from foci upon any tangent of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is b^2.
  - Tangents at the extremities of Latus Rectum pass through the corresponding foot of directrix on major axis.
  - Length of tangent b/w the point of contact and the point where it meets the directrix subtends right angle at the corresponding focus.

- Eqⁿ of Normal
  - Point form → $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$
  - Parametric form → $ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$

- Properties of normals
  - Normal other than major axis never passes through the focus.
  - Normal at the point P bisects angle SPS'
    [Reflection property]

- Co-normal Points: From any point in the plane maximum four normals can be drawn. Eccentric angle of all the four point, α, β, γ, δ then, $\alpha+\beta+\gamma+\delta = (2n+1)\pi$
- Concyclic Points: $\alpha+\beta+\gamma+\delta = 2n\pi$
- Eqⁿ of chord joining $P(\alpha)$ & $Q(\beta)$ → $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$
- PoI of tangents at $P(\alpha)$ & $Q(\beta)$ → $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

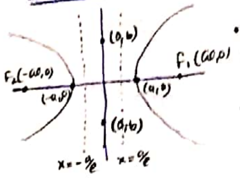
Hyperbola



$P_1F_2 - P_1F_1 = P_2F_1 - P_2F_2 = \text{const.}$

- **LR** $= \frac{2b^2}{a} = 2e(ae - a/e)$

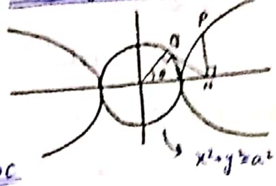
Standard Eqⁿ



Conjugate Hyperbola

- Hyperbola → $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \; (e_1)$
- conjugate Hyperbola → $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 (e_2)$
- $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$
- foci of the hyperbola and conj. are concyclic square.

Auxiliary Circle and Eccentric Angle



$Q(a\cos\theta, a\sin\theta)$
$P(a\sec\theta, b\tan\theta)$

COC

$T=0$

PoI of tangents from $P(\alpha)$ & $Q(\beta)$

$$\left(a\frac{\cos\frac{\alpha-\beta}{2}}{\cos\left(\frac{\alpha+\beta}{2}\right)}, b\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right)$$

Eqⁿ of chord joining $P(\alpha)$ and $Q(\beta)$

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

Pair of Tangents : $SS_1 = T^2$

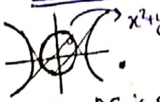
Hyperbolas reffered to two ⊥ lines :

$L_1 : lx + my + n = 0$
$L_2 : mx - ly + p = 0$

$$\frac{\left(\frac{lx+my+n}{\sqrt{l^2+m^2}}\right)^2}{a^2} - \frac{\left(\frac{mx-ly+p}{\sqrt{m^2+l^2}}\right)^2}{b^2} = 1$$

- Centre is poI of L_1 & L_2
- TA → 2a, CA → 2b
- TA is along $L_2 = 0$

Director Circle



$x^2+y^2=a^2-b^2$

- a>b, DC is real
- a=b, DC is point circle
- a<b, no real circle.

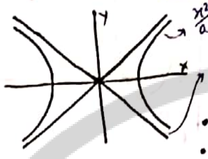
Eqⁿ of Tangent

- Point form : $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
- Parametric form; $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$
- slope form : $y = mx \pm \sqrt{a^2m^2 - b^2}$
- at (x_1,y_1) to $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$;
  $\frac{(x-h)(x_1-h)}{a^2} - \frac{(y-k)(y_1-k)}{b^2} = 1$

Chord with mp given

$T = S_1$

Asymptotes



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$y = \pm \frac{b}{a}x$

Important Points

- If angle b/w asymptotes of hyperbola is 2θ, $e = \sec\theta$
- Acute angle b/w asymptotes $\theta = \tan^{-1}\left|\frac{2ab}{a^2-b^2}\right|$
- Hyperbola and its conjugate have same asymptotes.
- Asymptotes pass thru the centre of the hyperbola
- The eqⁿ of pair of asymptotes differ from the eqⁿ of hyperbola ~~just~~ by only a constant.
- Asymptotes are diagonals of rectangle formed by lines drawn through extrimities of ~~the~~ each axis parallel to the other.
- For rectangular hyperbola, Asymptotes are at 90° i.e. $y = \pm x$
- At any point of a Asymptote if a st. line is drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted b/w the point and curve is always equal to the square of the semi conjugate axis.
- Perpendiculars from the foci on either asymptote meet it at the same point as the corresponding directrix and common points of intersections lie on aux circle.
- If the asymptotes of a rectangular hyperbola are $x = \alpha$ and $y = \beta$, then its eqⁿ is $(x-\alpha)(y-\beta) = c^2$

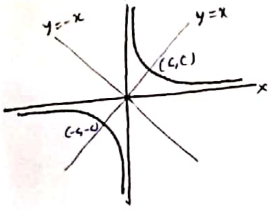
Equation of normal

- point form (x_1,y_1) : $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2+b^2$
- parametric form : $ax\cos\theta + by\cot\theta = a^2+b^2$

Properties of Normals

- Normal other than TA never passes through focus.
- Locus of feet of perpendicular drawn from focus upon any tanget of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is its aux circle i.e. $x^2+y^2=a^2$
- The product of perpendiculars drawn from foci upon any tangent of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is b^2
- The portion of the tangent b/w the poc and the point where it meets the directrix subtends a right angle at corresponding focus.
- The tangent and normal at any point of ~~contact~~ hyperbola bisect the angle b/w focal radii.
- If an ellipse and a hyperbola have same foci, the cut at right angles.
- The foci and the points P & Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter.

Rectangular Hyperbola



- Parametric form : $(ct, c/t)$
- Equation of tangent at 't' ; $x + yt^2 - 2ct = 0$
- Eqⁿ of Normal at 't' ; $xt^3 - yt - ct^4 + c = 0$
- Eqⁿ of tangent at (x_1,y_1) : $xy_1 + yx_1 = 2c^2$
- Eqⁿ of normal at (x_1,y_1) : $xx_1 - yy_1 = x_1^2 - y_1^2$

- $xy = c^2$, $e = \sqrt{2}$
- Asymptotes , $x=0, y=0$
- TA : $y=x$, CA: $y=-x$
- Vertex $A(c,c)$, $A'(-c,-c)$
- foci $(c\sqrt{2}, c\sqrt{2})$ & $(-c\sqrt{2}, -c\sqrt{2})$
- length of LR $= 2\sqrt{2}c$
- Aux circle → $x^2+y^2 = c^2$
- DC → $x^2+y^2=0$
- $x^2-y^2=1$ and $xy=1$ intersect at 90°

Concyclic points on $xy = c^2$

If a circle and a rectangular hyperbola $xy = c^2$ meets at four points $t_1\, t_2\, t_3\, t_4$, then,

- $t_1 t_2 t_3 t_4 = 1$
- Centre of the mean position of the four points bisects the distance b/w the centres of the two curves.

Theory of Equations and Logarithm

Laws of log

- $a^{\log_b x} = x^{\log_b a}$; $a, b > 0 \neq 1, x > 0$
- $\log_a x = \frac{1}{\log_x a}$
- $\log_a a = 1$, $\log_a 1 = 0$
- $\log_a x = \log_b x \cdot \log_a b = \frac{\log_b x}{\log_b a}$
- $\log_a (m^n) = n \log_a m$
- $\log_{a^n} (x) = \frac{1}{n} \log_a x$
- $\log_{a^n} x^m = \left|\frac{m}{n}\right| \log_a x$
- for $x > y > 0$

(i) $\log_a x > \log_a y$, if $a > 1$

(ii) $\log_a x < \log_a y$, if $0 < a < 1$

- $0 < a < 1$ then

(i) $\log_a x > p \Rightarrow 0 < x < a^p$

(ii) $0 < \log_a x < p \Rightarrow a^p < x < 1$

- $a > 1$,

(i) $\log_a x > p \Rightarrow x > a^p$

(ii) $0 < \log_a x < p \Rightarrow 0 < x < a^p$

Common Roots

- 1 common → $(Dur)^2 = Pass \cdot Pars$
- 2 common → $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Relation b/w roots and co-eff

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$

- $\sum \alpha_1 = -a_1/a_0$
- $\sum \alpha_1 \alpha_2 = a_2/a_0$ ……
- $\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$

Discriminant & Nature of Roots

$ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

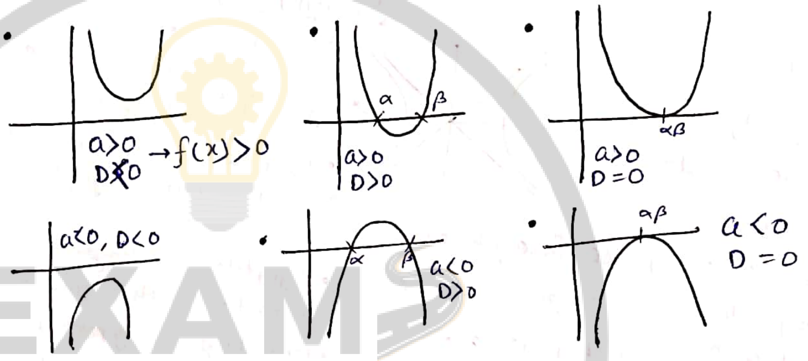
$\boxed{D = b^2 - 4ac}$

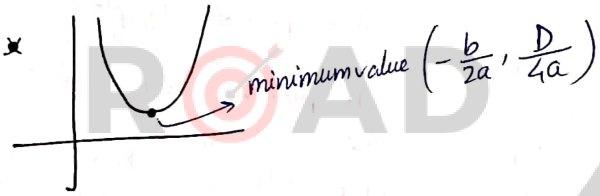
$D > 0$ → roots are real and distinct.

$D = 0$ → roots are real and equal

$D < 0$ → roots are imaginary.

- $f(x) = y = ax^2 + bx + c$





Binomial Theorem

- $(x+a)^n = {}^nC_0\,x^n + {}^nC_1\,x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots\cdots + {}^nC_r\,x^{n-r}a^r + \cdots\cdots + {}^nC_{n-1}\,x^1a^{n-1} + {}^nC_n x^0 a^n$

→ General Term: $T_{r+1} = {}^nC_r\,x^{n-r}a^r$

- $(x-a)^n = {}^nC_0x^n - {}^nC_1\,x^{n-1}a + {}^nC_2x^{n-2}a^2 - \cdots\cdots + (-1)^r\,{}^nC_nx^{n-r}a^r + \cdots\cdots + (-1)^n\,{}^nC_r\,x^{n-r}a^r$

→ General Term: $T_{r+1} = (-1)^r\,{}^nC_r\,x^{n-r}a^r$

- Middle term: (i) $\left(\frac{n}{2}+1\right)$th term, if n is even. $T_{\frac{n}{2}+1} = {}^nC_{n/2}\,x^{n/2}a^{n/2}$

  (ii) $\left(\frac{n+1}{2}\right)$th & $\left(\frac{n+3}{2}\right)$th term, if n is odd.

- Greatest term $\frac{T_{r+1}}{T_r} \geqslant 1$ i.e. $\frac{n-r+1}{r}\left|\frac{a}{x}\right| \geqslant 1$

- $\sum\limits_{r=0}^{n}(-1)^r\,{}^nC_r = 0$
- ${}^nC_1 - 2\,{}^nC_2 + 3\,{}^nC_3 - \cdots\cdots + n(-1)^{n-1}\,{}^nC_n = 0$
- ${}^nC_1 + 2\cdot{}^nC_2 + 3\,{}^nC_3 + \cdots\cdots + n\;{}^nC_n = n\cdot 2^{n-1}$
- $C_0C_r + C_1C_{r+1} + \cdots + C_{n-r}C_n = {}^{2n}C_{n-r}$
- ${}^nC_n + {}^{n+1}C_n + \cdots\cdots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

- Properties of Binomial Theorem
  - ${}^nC_r = \frac{n}{r}\;{}^{n-1}C_{r-1} = \frac{n}{r}\cdot\frac{n-1}{r-1}\cdot{}^{n-2}C_{r-2}$
  - ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
  - ${}^nC_r = {}^nC_{n-r}$
  - ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$
  - ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + \cdots = 2^{n-1}$

- Multinomial Theorem

$$(x_1+x_2+\cdots\cdots+x_k)^2 = \sum_{r_1+r_2+r_3+\cdots=n} \frac{n!\,x_1^{r_1}x_2^{r_2}\cdots x_k^{r_k}}{r_1!\,r_2!\cdots r_k!}$$

- no of terms in $(x+y+z)^n$ is ${}^{n+2}C_2$ or $\frac{(n+1)(n+2)}{2}$

Expressions

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\cdots + (-x)^r + \cdots \infty$
- $(1-x)^{-1} = 1 + x + x^2 + \cdots\cdots + x^r + \cdots\cdots \infty$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots\cdots + (r+1)(-x)^r + \cdots\cdots \infty$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\cdots + (r+1)\,x^r + \cdots \infty$
- $(1+x)^{-3} = 1 - 3x + 6x^2 - \cdots\cdots + \frac{(r+1)(r+2)}{2!}(-x)^r + \cdots \infty$
- $(1-x)^{-3} = 1 + 3x + 6x^2 + \cdots\cdots + \frac{(r+1)(r+2)}{2!}(x)^r + \cdots \infty$

Determinads

$$\Delta = \begin{array}{c|ccc|l} & C_1 & C_2 & C_3 & \\ & \downarrow & \downarrow & \downarrow & \\ & a_{11} & a_{12} & a_{13} & \leftarrow R_1 \\ & a_{21} & a_{22} & a_{21} & \leftarrow R_2 \\ & a_{31} & a_{32} & a_{33} & \leftarrow R_3 \end{array}$$

[3×3]

$a_{\underset{\text{row}}{m}\underset{\text{column}}{n}}$

Minor

- Minor of $a_{11}, M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$
- Minor of $a_{22}, M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13}$
- Minor of $a_{33}, M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{23} & a_{22} \end{vmatrix} = a_{12}a_{23} - a_{12}a_{21}$

Co-factor

co-factor of $a_{ij} = (-1)^{i+j} M_{iy}$

o-factor of $a_{11} = (-1)^{1+1} M_{11} = M_{11}$

co-factor of $a_{12} = (-1)^{1+2} M_{12} = -M_{12}$

Signs of cofactors

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Expansion of Determinads

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Properties

$$\begin{vmatrix} 0 & a & b \\ 0 & c & d \\ 0 & e & f \end{vmatrix} = 0$$

ny row/coloumn 0

- $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

Whenever we interchange any two row (or coloumn) value of it will be multiplied by '-ve'

- $\begin{vmatrix} a+b & c & d \\ e+f & g & h \\ i+j & k & l \end{vmatrix} = \begin{vmatrix} a & c & d \\ e & g & h \\ i & k & l \end{vmatrix} + \begin{vmatrix} b & c & d \\ f & g & h \\ j & k & l \end{vmatrix}$

- $\begin{vmatrix} ap & bp & cp \\ d & e & f \\ g & h & i \end{vmatrix} = p \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

Any two rows or column same.

- Transform

$\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ $R_1 = R_1 + pR_2$

$\Delta = \begin{vmatrix} a+dp & b+ep & c+fp \\ d & e & f \\ g & h & i \end{vmatrix}$

$$\begin{vmatrix} ap & d & g \\ bp & e & h \\ cp & f & i \end{vmatrix} = p \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\rightarrow a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 0$

or

$a_{11}a_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$

$a_{12}C_{13} + a_{22}a_{23} + a_{32}a_{33} = 0$

Important Expansion

- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$
- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$
- $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+c$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

System of Evalute

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_2 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Note

- If $[\Delta \neq 0]$.
  - → Unique Sol^n^,
  - → Consistent set of solⁿ

- If $[\Delta = 0]$
  - → (a) if any one (or two) of $\Delta_x, \Delta_y, \Delta_z$ is/are Non-zero, Inconsistent System ⇒ No solⁿ
  - → (b) $[\Delta_x = \Delta_y = \Delta_z = 0]$ Consistent set of solⁿ ⇒ Infinite no of colⁿ.

$x = \frac{\Delta_x}{\Delta}$

$y = \frac{\Delta_y}{\Delta}$

$z = \frac{\Delta_z}{}$

Cramer's Rule ($\Delta \neq 0$)

Trigonometric Equation

- $\sin\theta = 0 \longrightarrow \theta = n\pi,\ n\in Z$
- $\cos\theta = 0 \longrightarrow \theta = (2n+1)\frac{\pi}{2},\ n\in Z$
- $\tan\theta = 0 \longrightarrow \theta = n\pi,\ n\in Z$
- $\sin\theta = 0 \longrightarrow \theta = (4n+1)\frac{\pi}{2},\ n\in Z$
- $\sin\theta = -1 \longrightarrow \theta = (4n-1)\frac{\pi}{2},\ n\in Z$
- $\cos\theta = 1 \longrightarrow \theta = 2n\pi,\ n\in Z$
- $\cos\theta = -1 \longrightarrow \theta = (2n+1)\pi,\ n\in Z$
- $\cot\theta = 0 \longrightarrow \theta = (2n+1)\frac{\pi}{2},\ n\in Z$

- $\sin\theta = \sin\alpha \longrightarrow n\pi + (-1)^n\alpha,\ n\in Z$

$\Rightarrow \sin\theta = k \longrightarrow \theta = n\pi + (-1)^n(\sin^{-1}k),\ n\in Z$, $k\in[-1,1]$

- $\cos\theta = \cos\alpha \longrightarrow \theta = 2n\pi \pm \alpha,\ n\in Z$

$\Rightarrow \cos\theta = k \longrightarrow \theta = 2n\pi \pm (\cos^{-1}k),\ n\in Z,\ k\in[-1,1]$

- $\tan\theta = \tan\alpha \longrightarrow \theta = n\pi + \alpha,\ n\in Z$

$\Rightarrow \tan\theta = k \longrightarrow \theta = n\pi + (\tan^{-1}k),\ k\in R$

- $\sin^2\theta = \sin^2\alpha \ /\ \cos^2\theta = \cos^2\alpha$

$\longrightarrow \theta = n\pi \pm \alpha,\ n\in Z$

- $\tan^2\theta = \tan^2\alpha \longrightarrow \theta = n\pi \pm \alpha,\ n\in Z$

- Solution of the equation of the form $a\cos\theta + b\sin\theta = c$

$\longrightarrow$ If $|c| > \sqrt{a^2+b^2}$, then no real solution

$\longrightarrow$ If $|c| \le \sqrt{a^2+b^2}$, then divide both sides of the equation by $\sqrt{a^2+b^2}$, then take $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$,

$\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$, equation will reduce to

$\cos(\theta-\alpha) = \cos\beta$, where $\tan\alpha = b/a$

$\cos\beta = \frac{c}{\sqrt{a^2+b^2}}$

• If we take $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$,

$\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$, then the equation will reduce to $\sin(\theta+\alpha) = \sin\beta$,

$\sin\beta = \frac{c}{\sqrt{a^2+b^2}}$

- While solving trigo equation, avoid squaring the equation as far as possible. If squaring is necessary check the solution for extraneous values (similar values following the same pattern.

- Never cancel terms containing unknown terms on the two sides which are in product. It may cause the loss of a genuine solution.

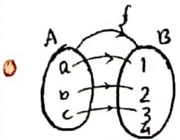
- The answer should not contain such values of angles which make any term undefined or infinite.

- Domain should not change while simplifying the equation. If it changes, necessary corrections must be made.

- Check the denominator is not zero at any stage while solving the equation.

- Extreme values of functions ↶ Keep in mind.

Functions



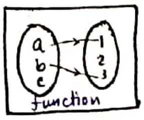
$f: A \to B \equiv ((a,1),(b,2),(c,3))$

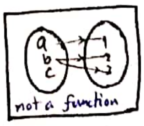
Domain of $f: \{a, b, c\}$ → Input

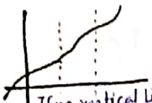
Range of $f: \{1,2,3\}$ → output

Co-domain of $f: \{1,2,3,4\}$ → possible outcome

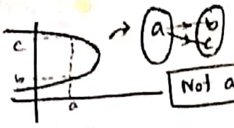
Condition: A mapping is a function in each ~~ead~~ input have one and only one outputs.





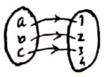


If no vertical line cuts the graph more than once, it a function



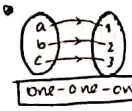
Not a function

Types of mapping/ functions:

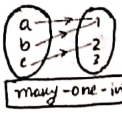


Range ⊂ cordomain

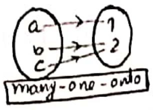
$\boxed{\text{One-one-into}}$



$\boxed{\text{one-one-onto}}$



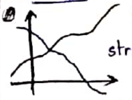
$\boxed{\text{many-one-into}}$



$\boxed{\text{many-one-onto}}$

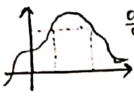
- one-one → Injection
- Onto → Surjection
- one-one-onto → Bijection

slope:



strictly increasing $(\frac{dy}{dx} > 0)$

strictly decreasing $(\frac{dy}{dx} < 0)$



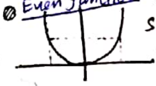
$\frac{dy}{dx} = 0$ (for real x)

$\boxed{\text{many-one}}$

If a horizontal line cuts the graph at more than one point, its a many-one or else one-one

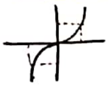
Type of function:

Even function If $f(-x) = f(x)$



Symmetric about y axis

Odd function: If $f(-x) = -f(x)$

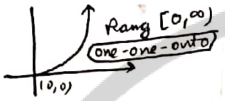


Symmetric about origin.

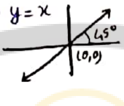
Fundamental graphs:

- $f: R \to R : f(x) = x^2$

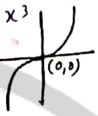
- $f: [0, \infty) \to [0, \infty) : f(x) = x^2$



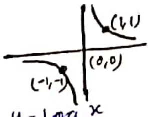
- $y = x$



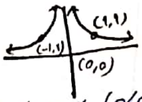
- $y = x^3$



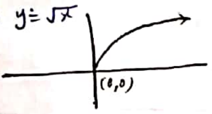
- $y = 1/x$ / $xy = c^2$



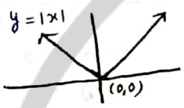
- $y = 1/x^2$



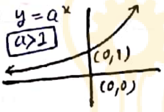
- $y = \sqrt{x}$



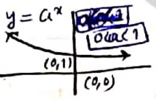
- $y = |x|$



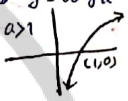
- $y = a^x$ $\boxed{a>1}$



- $y = a^x$ ~~0<a<1~~ $\boxed{0<a<1}$

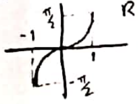


- $y = \log_a x$ $a>1$



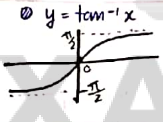
- $y = \log_a x$ $(0<a<1)$

- $y = \sin^{-1}x$ Domain$[-1,1]$ R$[-\frac{\pi}{2}, \frac{\pi}{2}]$ $\boxed{\text{odd}}$

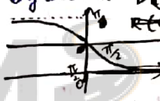


- $y = \cos^{-1}x$ D$[-1,1]$ R$[0,\pi]$

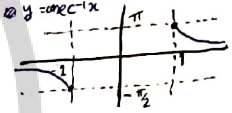
- $y = \tan^{-1}x$ D$(-\infty,\infty)$ R$(-\frac{\pi}{2}, \frac{\pi}{2})$



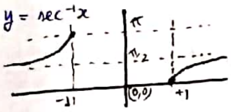
- $y = \cot^{-1}x$ D$(-\infty, \infty)$ ~~R(0 Rang~~ R $(0, \pi)$



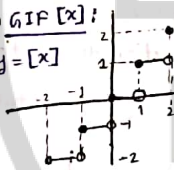
- $y = \text{cosec}^{-1}x$



- $y = \sec^{-1}x$



- GIF $[x]$: $y = [x]$



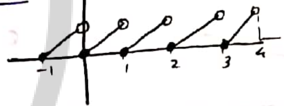
$\boxed{(x-1) < [x] \leq x}$

$[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \in I \end{cases}$

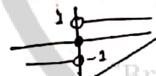
- Fractional part of x: $\{x\}$

$\{x\} = x - [x]$

$0 \leq \{x\} < 1$

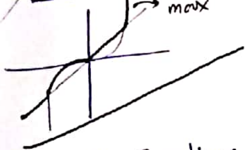


Signum Function:

$\text{sgn}(x) = \frac{|x|}{x} = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$

Graphical Transformation:

- $y = f(x) \to y = -f(x)$: mirror img about x axis.
- $y = f(x) \to y = f(-x)$: mirror img about y axis
- $y = f(x) \rightsquigarrow f(x \pm k)$: $-k$ → right/ $+k$ → left
- $y = f(x) \rightsquigarrow f(x) \pm k$: $+k$ → up / $-k$ → down.
- $y = f(x) \rightsquigarrow |f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$
- $f(x) \rightsquigarrow y = f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$

min/max function:



Inverse function:

$f(x)$ is invertible only if it is one-one-onto / Bijective.

$f^{-1}(f(x)) = x$ | $[f'(f(x))]' f'(x) = 1$ | $f^{-1}(f(x))' = \frac{1}{f'(x)}$ Put $x = 1$, $y = k$ (say)

$\boxed{y^2 - xy^3 - x^3 = 0}$ → Implicite function

Periodic Function:

If $f(x+T) = f(x)$, $T > 0$ → $f(x)$ is called period function with period T.

smallest value of T is called fundamental period of $f(x)$

→ period of $\sin x, \cos x, \text{cosec}x, \sec x \to 2\pi$ | → $\tan x$, $\cot x \to \pi$ | $|\sin x|, |\cos x| \ldots \to \pi$

→ $\sin^n x, \cos^n x, \sec^n x, \text{cosec}^n x \longrightarrow \begin{cases} 2\pi, & n \in \text{odd} \\ \pi, & n \in \text{even} \end{cases}$ | $\tan^n x$, $\cot^n x \to \pi$ | If $f(x) \to T \rightsquigarrow f(x) \pm k \to T$ $\rightsquigarrow f(x \pm k) \to T \rightsquigarrow k f(x) \to T$

→ const functions are periodic but period not defined. | $f(x) \to T_1$, $g(x) \to T_2$ → $f(x) \pm g(x) / \frac{f(x)}{g(x)}$ / $f(x) \cdot g(x) \to \text{LCM}(T_1, T_2)$

→ $f(x-a) = f(x+a) \to T = 2a$

→ $f(a-x) = f(a+x) \to f(x)$ is symmetric abt. line $x=a$ ~~period 2|a-b|~~

→ $f(a-x) = f(a+x)$ & $f(b-x) = f(b+x)$ → Periodic abt. $2|a-b|$

→ $f(a-x) = f(a+x) \to 2|a-b|$, $f(b-x) = f(b+x) \to 2|b-c|$, $f(c-x) = f(c+x)$ → $2|a-c|$

Period of $f(x) = \min\{2|a-b|, 2|b-c|, 2|c-a|\}$

Limits

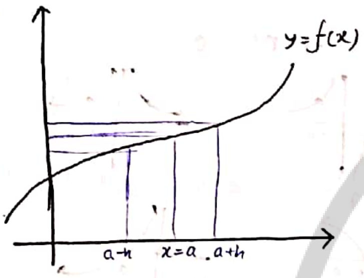
Expansions:

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
- $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5$
- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$
- $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$
- ~~log~~ $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $a^x = 1 + \frac{x\ln a}{1!} + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \cdots$
- $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$
- $\sin^{-1}x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots$

Important Results

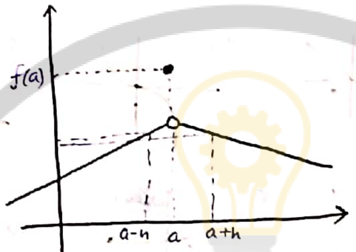
- $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{x}{\tan x} = 1$
- $\lim_{x\to 0} \frac{\sin^{-1}x}{x} = \lim_{x\to 0} \frac{\tan^{-1}x}{x} = 1$
- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$
- $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Continuity

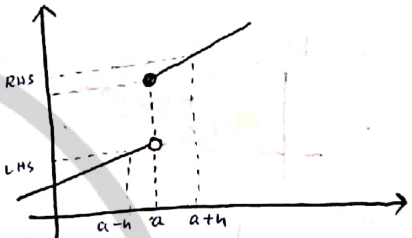


If $\underset{x=a}{LHL} = \underset{x=a}{RHL} = f(a)$

→ $y = f(x)$ is continuous at $x = a$

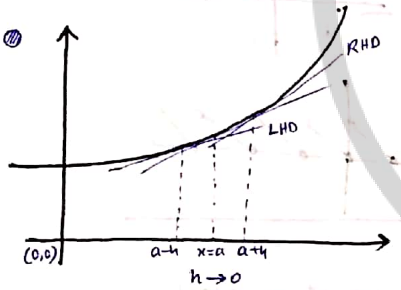


If $LHL = RHL \neq f(a)$, Discontinuous at $x = a$, point discontinuity / Removable discontinuity.



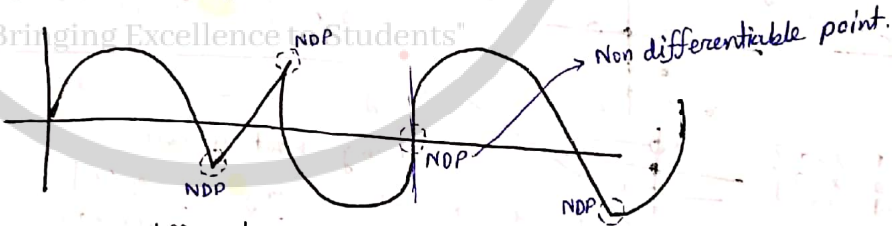
$LHL \neq RHL$, Discontinuous at $x = a$. Jump Discontinuity.

Differentiability



If $LHD = RHD$ at $x = a$, $f(x)$ is differentiable at $x = a$

- Sharp turns lead to non-differentiable points.
- Smooth curves are generally differentiable at all points.
- Tangents must have finite slope to make function differentiable.



RHD at $x=a$

$$\boxed{Rf'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$$

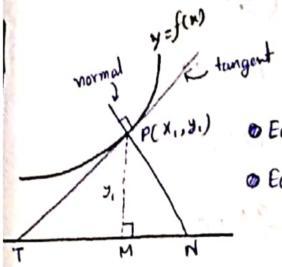
LHD at $x=a$

$$\boxed{Lf'(a) = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}}$$

- Discontinuous ⇒ non-differentiable
- Differentiable ⇒ continuous.

$f(x) \to$ diff $\to f'(x) \to$ cont / $f''(x) \to$ cont $\to f'(x) \to$ diff

Application of Derivatives



- Slope of tangent $= \frac{d}{dx} f(x)\Big|_{(x_1,y_1)} = \tan\theta$
- Slope of normal $= -\frac{dx}{dy}\Big|_{(x_1,y_1)} = -\cot\theta$
- Equation of tangent: $(y-y_1) = \frac{dy}{dx}\Big|_{(x_1,y_1)} (x-x_1)$
- Equation of normal: $(y-y_1) = -\frac{dx}{dy}\Big|_{(x_1,y_1)} (x-x_1)$

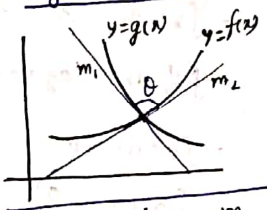
length of tangent(PT) $= |y|\operatorname{cosec}\theta$, $\boxed{|PT| = |y_1|\sqrt{1+\left(\frac{dx}{dy}\right)^2}}$

length of normal (PN) $= |y_1|\sec\theta$, $\boxed{|PN| = |y_1|\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$

length of Sub-tangent $= |y_1|\cot\theta$ $\boxed{|TM| = |y_1|\left|\frac{dx}{dy}\right|}$

Length of Sub-normal $= |y_1|\tan\theta$ $\boxed{|MN| = |y_1|\left|\frac{dy}{dx}\right|}$

Angle b/w 2 curves



$\frac{d}{dx} f(x)\Big|_{(x_1,y_1)} = m_1$

$\frac{d}{dx} g(x)\Big|_{(x_1,y_1)} = m_2$

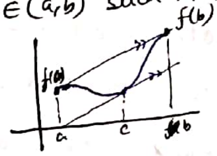
$\boxed{\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1+m_1 m_2}\right|}$

- $\theta = 90°$ is called orthogonal intersection.

Lagranges Mean Value Theorem (LMVT)

if $y=f(x)$ is continuous on $[a,b]$ & differentiable on (a,b), There exist at least one such value of $c \in (a,b)$ such that

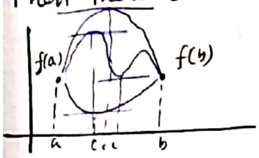
$\boxed{f(c) = \frac{f(b)-f(a)}{b-a}}$



Rolle's Theorem

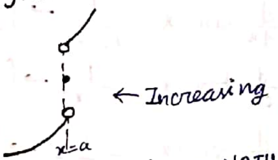
$f(x)$ is cont on $[a,b]$, diff on (a,b) and $f(a) = f(b)$

Then there exist at least one $c \in (a,b)$ so that $f'(c) = 0$



Monotonocity:

- $\frac{dy}{dx} > 0 \rightarrow y=f(x)$ is an increasing function
- $\frac{dy}{dx} < 0 \rightarrow y=f(x)$ is an strictly decreasing function
- $\frac{dy}{dx} \geqslant 0 \rightarrow y=f(x)$ is an non-decreasing function
- $\frac{dy}{dx} \leqslant 0 \rightarrow y=f(x)$ is a non-increasing function.



Maxima / Minima:

$\frac{dy}{dx} = 0 \ , \Rightarrow x = a, b$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$\frac{d^2y}{dx^2}\Big|_{x=a} < 0$ [x=a is maxima]

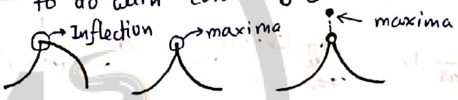
$\frac{d^2y}{dx^2}\Big|_{x=b} > 0$ [x=b is minima]

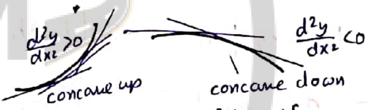
$\frac{d^2y}{dx^2}\Big|_{x=c} = 0$ [x=c is inflection]

For Inflection:

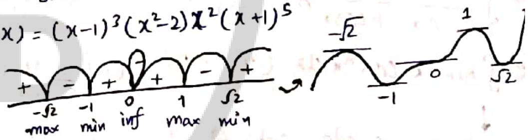
- $\frac{dy}{dx}$ NEED NOT TO BE ZERO
- $\frac{d^2y}{dx^2} = 0$
- Around point of inflection, graph changes its concavity.

☆ Monotonocity/Maxima/Minima have NOTHIN to do with continuity of the graph.





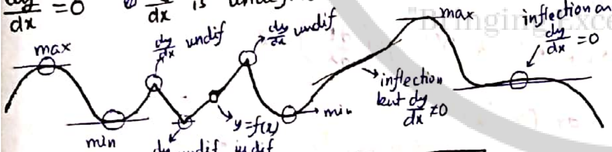
Q» $f'(x) = (x-1)^3(x^2-2)x^2(x+1)^5$

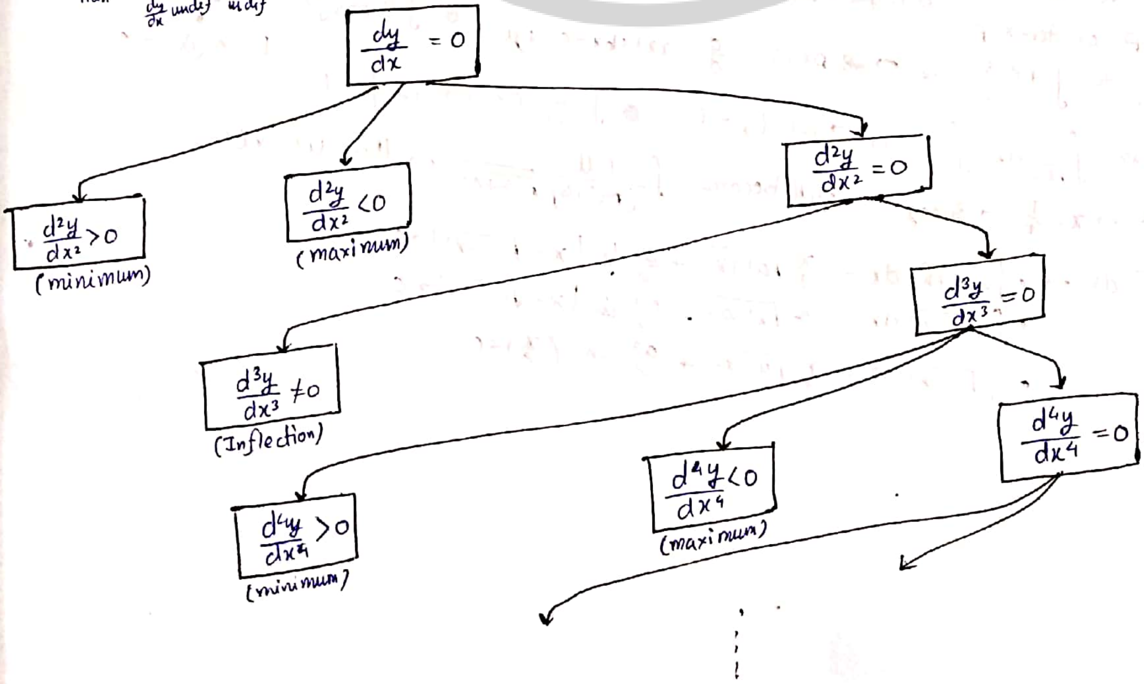


- Note: $\frac{d^2y}{dx^2} = 0$ at $x=a$ is a point of inflectio provided $\frac{d^3y}{dx^3}$ is non zero at $x=a$

Critical points:

$\frac{dy}{dx} = 0$ • $\frac{dy}{dx}$ is undefined or • $y=f(x)$ is undefined





Indefinite Integral

- $\frac{d}{dx} x^n = nx^{n-1} \rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c$
- $\frac{d}{dx} \log x = \frac{1}{x} \rightarrow \int \frac{1}{x} dx = \log x + c$
- $\frac{d}{dx} e^x = e^x \longrightarrow \int e^x dx = e^x$
- $\frac{d}{dx} a^x = a^x \ln a \longrightarrow \int a^x dx = \frac{a^x}{\ln a} + c$
- $\frac{d}{dx} \sin x = \cos x \longrightarrow \int \cos x \, dx = \sin x + c$
- $\frac{d}{dx} \cos x = -\sin x \longrightarrow$ ~~∫integral~~ $\int \sin x \, dx = -\cos x + c$
- $\frac{d}{dx} \tan x = \sec^2 x \longrightarrow \int \sec^2 x \, dx = \tan x + c$
- $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x \longrightarrow \int \operatorname{cosec}^2 x \, dx = -\cot x + c$
- $\frac{d}{dx} \sec x = \sec x \tan x \longrightarrow \int \sec x \tan x \, dx = \sec x + c$
- $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x \longrightarrow \int \operatorname{cosec} x \cot x \, dx = \operatorname{cosec} x + c$
- $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \rightarrow \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
- $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \rightarrow \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
- $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \rightarrow \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin$
- $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}(x/a) + c$
- $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1}(x/a) + c$
- $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$
- $\int \tan x \, dx = \log|\sec x| + c$
- $\int \cot x \, dx = \log|\sin x| + c$
- $\int \sec x \, dx = \log|\sec x + \tan x| + c$
  $= \log \left| \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \right| + c$
- $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$
  $= \log \left| \tan \frac{x}{2} \right| + c$

- ILATE
  Inverse
  log
  Algebraic
  Trigo
  expo

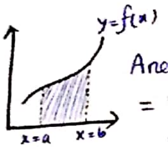
whichever comes first is the Ist function is by parts

- ~~$\frac{d}{dx}$~~
- **By Parts:** $\int I \cdot II \, dx = I \int II \, dx - \int \left(\left(\frac{d}{dx} I\right) \left(\int II \, dx \right) \right) dx$
- $\int e^x (f(x) + f'(x)) = e^x f(x) + c$

Forms

- $\int \frac{1}{\text{linear}} dx = \frac{\log|\text{linear}|}{\text{coeff of } x} + c$
- $\int \frac{1}{(\text{linear})^n} dx = \int \frac{(\text{linear})^{n+1}}{(-n+1)(\text{coeff of } x)} + c$
- $\int \log x \, dx = x \log x - x + c$
- $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$
- $\int (f(x))^n f'(x) \, dx = \frac{(f(x))^{n+1}}{n+1} + c$
- $\int \frac{dx}{\text{Quad}}$
  - $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}(x/a)$
  - $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$
  - $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$
- $\int \frac{1}{a \sin x + b \cos x} dx \rightarrow$ put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$, $\cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$
- $\int \sin^m x \cos^n x \, dx \ (m, n \in N)$
  $\downarrow$
  - If $m, n \in$ odd, subs ~~at~~ any
  - If one is odd, sub even
  - If both are even, use trigo
  - If both are rational and $\frac{m+n-2}{2}$ is $-$ve int. then sub $\cot x = p$ or $\tan x = p$
- $\int \frac{dx}{a\cos^2 x + b\sin^2 x}$, $\int \frac{dx}{a + b\sin^2 x}$, $\int \frac{dx}{(a \sin x + b\cos x)^2}$, $\int \frac{dx}{a + b\sin^2 x + \cos^2 x} \rightarrow$ divide N^r & D^r by $\cos^2 x$
- $\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c} dx \rightarrow$ WR, $N^r = \lambda(D^r) + \mu\left(\frac{dD^r}{dx}\right) + \gamma$
- **Biquadratic** $\rightarrow$ sub $(x + 1/x)$ or $(x - 1/x) = t$
- $\int \text{linear} \sqrt{\text{Quad}} \rightarrow L = m(Q)' + n$
- $\int \frac{px+q}{ax^2+bx+c} dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx \rightarrow$ WR $px + q = \frac{d}{dx}(ax^2+bx+c) + \mu$
- $\int \frac{1}{L_1\sqrt{L_2}} dx$, $\int \frac{L_1}{\sqrt{L_2}} dx$, $\int \frac{\sqrt{L_2}}{L_1} dx \rightarrow$ sub $L_2 = t^2$
- $\int \frac{1}{L\sqrt{Q}} dx \rightarrow$ sub $\frac{1}{t} = L$
- $\int \frac{1}{Q_1\sqrt{Q_2}} dx \rightarrow x = \frac{1}{t} \rightarrow$ Integrand will become $\int \frac{t\,dt}{(pt^2+q)\sqrt{(rt^2+s)}} \rightarrow$ then, $u^2 = rt^2 + s$
- $\int \sqrt{\text{Quad}} \, dx \rightarrow$
  - $\int \sqrt{a^2+x^2} \, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} \ln|x + \sqrt{a^2+x^2}| + c$
  - $\int \sqrt{x^2-a^2} \, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2-a^2}| + c$
  - $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}(x/a) + c$

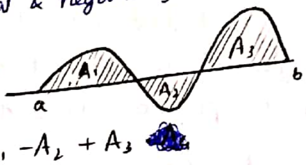
DEFINITE INTEGRATIONS



Area of shaded region $= \int_a^b f(x)\,dx$

- $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

- Region lying above x axis will give +ve value of integral & negative for the portion lying below x axis.



$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$

- **Properties:**
  - $\int_a^b f(x)dx = \int_a^b f(t)\,dt$
  - $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ [c may or may not belong to (a,b)]

- $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ [Turning Property]

- $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

- $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even func i.e. } f(-x)=f(x) \\ 0, & \text{if } f(x) \text{ is odd func} \end{cases}$

- While substituting, t must be conteneous in the interval

- **Properties related to periodic func:** [if $f(x+T)=f(x)$, period is T]

- $\int_{a+nT}^{a+nT} f(x)\,dx = \int_a^b f(x)\,dx$

- $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$, $n \in I$

- $\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx$, $n \in I$

- $\boxed{\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \cos x\,dx = -\frac{\pi}{2}\log 2}$

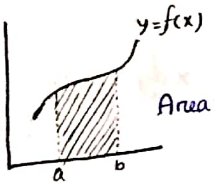
- $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx$, $n, m \in I$

- **Newton-Leibnitz 1st Rule:** $\boxed{\frac{d}{dx}\left(\int_{f(x)}^{g(x)} h(t)\,dt\right) = h(g(x)) \times \frac{d}{dx}(g(x)) - h(f(x)) \times \frac{d}{dx}(f(x))}$

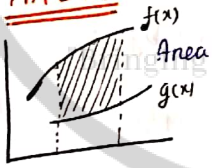
Only for IIT ⚠

- **Leibnitz 2nd Rule:** $\boxed{\text{If } I(d) = \int_a^b f(x,d)\,dx \rightarrow \cancel{\frac{\partial I}{\partial}}\ \cancel{\frac{dI}{d}}\ \frac{\partial I}{\partial d} = \int_a^b \frac{\partial f(x,d)}{\partial d}\,\cancel{dx}}$

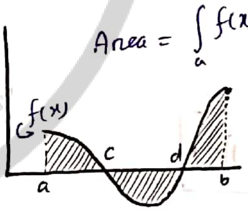
AREA

- 

Area $= \int_a^b f(x)\,dx$

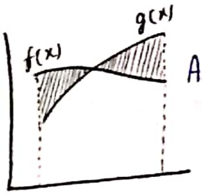
- 

Area $= \int_a^b (f(x) - g(x))\,dx$

- 

Area $= \int_a^c f(x)\,dx + \left|\int_c^b f(x)\,dx\right| + \int_d^b f(x)\,dx$

→ c and d are roots of $f(x)=0$

- 

Area $= \int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx$

- **Vertical Strip:**

Area $= \int_{x=a}^{x=b} (\text{upper } y - \text{lower } y)\,dx$

- **Horizontal Strip:**

Area $= \int_{y=c}^{y=d} (\text{Right } x - \text{left } x)\,dy$

DIFFERENTIAL EQUATION

- Eq involving x, y & differentiate co-efficient.
- DE represents a family of curves.
- **Order:** Order of highest order derivative present in the eqⁿ is the order of D.E.
- **Degree:** Degree of the highest order derivative present in the eqⁿ is the degree of DE, provided the eqⁿ is polynomial in different co-eff and eqⁿ is free from radicals.

Formation of DE:

(Degree of a DE = No of arbitrary constants present in eqⁿ)

DE of all lines passing thru origin:

$y = mx \qquad \frac{dy}{dx} = m \nearrow y = \frac{dy}{dx}x \rightarrow \boxed{x\,dy - y\,dx = 0}$

DE of all lines: $y = mx + c$

$\frac{dy}{dx} = m \;,\; \boxed{\frac{d^2y}{dx^2} = 0}$

Solution of DE:

- **Variable-seperable form:** $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$\Rightarrow \frac{dy}{dx} = \frac{e^x + x^2}{e^y} \Rightarrow \int e^y dy = \int (e^x + x^2) dx$

- **Eqⁿ Reducible to Variable Separable form**

$\frac{dy}{dx} = f(ax + by + c)$, consider, $ax + by + c = t$

- **Homogeneous form:**

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are of same order.

$\frac{dy}{dx} = h(y/x)$ assume $y/x = t$

- **Eqⁿ reducible to Homogenous form:** $\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + D}$

- [If $aB \neq Ab$ or $A + b \neq 0$]

$x = X + h \qquad y = Y + k$

$dx = dX \qquad dy = dY$

$\therefore \frac{dY}{dX} = \frac{aX + bY + ah + bk + c}{AX + BY + Ah + Bk + D} \nearrow \left.\begin{matrix} ah + bk + c = 0 \\ \& \\ Ah + Bk + D = 0 \end{matrix}\right] \nearrow$ find value of h & k

$\frac{dY}{dX} = \frac{aX + bY}{AX + BY} \rightarrow$ Homogeneous $\longrightarrow$ In the end, $X = x - h$, $Y = y - k$

- If $aB = Ab \longrightarrow \overset{\text{Assume}}{(ax + by = t)}$
- If $A + b = 0 \rightarrow$ simply cross multiply & replace $x\,dy + y\,dx$ by $d(xy)$

Linear Differential Eqⁿ:

$\Rightarrow \frac{dy}{dx} + Py = Q$ [P & Q are func of x alone]

$I.F = e^{\int P dx}$

$\rightarrow \boxed{y(I.F) = \int (Q(I.F)) dx}$

$\Rightarrow \frac{dx}{dy} + Mx = N$ [M & N are func of y alone]

$I.F. = e^{\int M dy}$

$\rightarrow \boxed{x(I.F.) = \int (N(I.F)) dy}$

Barnoulli Eqⁿ

$\frac{dy}{dx} + \frac{y}{x} = y^n$

divide by y^n and then assume ~~B~~-1 co-eff of x as t

here, $t = \frac{1}{y^{n-1}}$

curves.

gree of DE, provided the
ls.

of all lines: $y = mx+c$

$\frac{dy}{dx} = m$, $\boxed{\frac{d^2y}{dx^2} = 0}$

le Seperable form:

consider, $ax+by+c = t$

$\frac{ax+by+c}{Ax+By+D}$

$+C=0$] find value of h & k
$k+D=0$]

→ In the end, $X = x-h$
$Y = y-k$

& replace $xdy+ydx$ by $d(xy)$

coeff of x as t.

VECTORS

Angle bisector b/w two vectors:

Internal → $\vec{R} = \lambda(\hat{a}+\hat{b})$

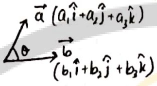
External → $\vec{Q} = \mu(\hat{a}-\hat{b})$

Section formula:

Internal → $\left(\frac{m\vec{b}+n\vec{a}}{m+n}\right)$

External → $\left(\frac{m\vec{b}-n\vec{a}}{m-n}\right)$

Dot (Scalar) Product:



- $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
- $\vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
- $\vec{a}\cdot\vec{b} = 0$ → perpendicular
- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$
  $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- Angle b/w the vectors →

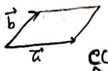
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

- Projection of $\vec{a}$ on $\vec{b}$ →

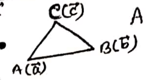
$$\vec{P} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

Cross (Vector) Product:

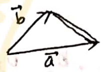
- $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{r}$
- $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- 

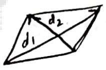
Area = $|\vec{a}\times\vec{b}|$

- 

$A = \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

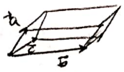
- 

Area $= \frac{1}{2}|\vec{a}\times\vec{b}|$

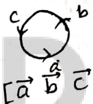
- 

Area $= \frac{1}{2}|\vec{d_1}\times\vec{d_2}|$

Scalar Triple product / Box Product: $[\vec{a}\;\vec{b}\;\vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$ → volume of parallelop—



- $[k\vec{a}\;\vec{b}\;\vec{c}] = k[\vec{a}\;\vec{b}\;\vec{c}]$
- $[\vec{a}\vec{b}\vec{c}] = [\vec{b}\;\vec{c}\;\vec{a}] = [\vec{c}\;\vec{a}\;\vec{b}]$
- $[\vec{a}\;\vec{b}\;\vec{c}] = -[\vec{b}\;\vec{a}\;\vec{c}]$



- $[\vec{a}\;\vec{c}\;\vec{d}] + [\vec{b}\;\vec{c}\;\vec{d}]$
  $= [\vec{a}+\vec{b}\;\;\vec{c}\;\;\vec{d}]$
- $[\vec{a}\;\vec{b}\;\vec{b}] = 0$
- $[\vec{a}\;\vec{b}\;\vec{c}] = 0$ ↗ $\vec{a}, \vec{b}, \vec{c}$ are coplanar.
- $[\vec{a}\;\vec{b}\;\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Vector Triple Product:

- $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\cdot\vec{a}$
- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

: 3D Geometry :

Direction Cosines:

$P(x, y, z)$ $\alpha, \beta, \gamma \rightarrow$ direction angles

$\cos\alpha, \cos\beta, \cos\gamma$ → direction cosines

$\cos\alpha = l, \ \cos\beta = m, \ \cos\gamma = n$

$\cos\alpha = \frac{x}{r}, \ \cos\beta = \frac{y}{r}, \ \cos\gamma = \frac{z}{r}$

$\boxed{l^2 - m^2 + n^2 = 1}$

Direction Ratios: Simple ratio of DC.

DR → DC >> $DR(a, b, c) \rightarrow DC\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

DC → DR >> $DC\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \rightarrow DR(1, -1, 1)$ or $(2, -2, 2)$ or $(\lambda, -\lambda, \lambda)$

- (a_1, b_1, c_1) & $(a_2, b_2, c_2) \rightarrow \cos\theta = \frac{a_1a_2 - b_1b_2 + c_1c_2}{\sqrt{a_1^2 - b_1^2 + c^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$ → $= 0$ if $\theta = 90°$

(l_1, m_1, n_1) & $(l_2, m_2, n_2) \rightarrow \cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

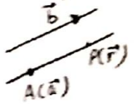
- (l_1, m_1, n_1) & (l_2, m_2, n_2) DC of 2 vectors → internal bisector $(l_1 + l_2, m_1 + m_2, n_1 + n_2)$, external bisector $(l_1 - l_2, m_1 - m_2, n_1 - n_2)$
- DR of line joining $A(a_1, b_1, c_1)$ & $B(a_2, b_2, c_2) \rightarrow (a_1 - a_2, b_1 - b_2, c_1 - c_2)$

	2D	3D
$x = 0$	y axis	yz plane
$y = 0$	x axis	xz plane
$x = 0, y = 0$	origin	z axis

- In 3D line is the intersection of 2planes.

Equation of line in 3D:

① Equation of line passing thru a point $\vec{a}$ and parallel to another vector $\vec{b}$



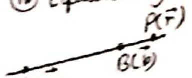
$\boxed{\vec{r} = \vec{a} + \lambda\vec{b}}$ vector form

⑪ $\vec{r} \equiv (x, y, z)$, $\vec{a} = (x_1, y_1, z_1)$, $\vec{b}$ (DR) $\equiv (a, b, c) \rightarrow a\hat{i} + b\hat{j} - c\hat{k}$

$\vec{r} - \vec{a} = \lambda\vec{b}$

$\boxed{\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda}$ Cartesian form

⑪ Equation of line passing thru 2 points.



$\boxed{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})}$ vector form

$$\boxed{\begin{array}{c}\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda \\ \text{or} \\ \frac{x - x_2}{x_2 - x_1} = \frac{y - y_2}{y_2 - y_1} = \frac{z - z_2}{z_2 - z_1} = \lambda\end{array}}$$ Cartesian form

Angle b/w 2 lines: $\vec{r} = \vec{a} + \lambda\vec{b}$, $\vec{r} = \vec{c} + \mu\vec{d}$

$\cos\theta = \frac{\vec{b} \cdot \vec{d}}{|\vec{b}||\vec{d}|}$

Shortest distance b/w 2Lines: [Shortest dist = 0 if intersecting]

→ If parallel → $\vec{r} = \vec{a} + \lambda\vec{b}$, $\vec{r} = \vec{c} + \mu\vec{b}$

→ then shortest dist $= \frac{|(\vec{a} - \vec{c}) \times \vec{b}|}{|\vec{b}|}$

→ If skew: → $\vec{r} = \vec{a} + \lambda\vec{b}$, $\vec{r} = \vec{c} + \mu\vec{d}$

Shortest dist $= \frac{[(\vec{a} - \vec{c})\,\vec{b}\,\vec{d}]}{|\vec{b} \times \vec{d}|}$

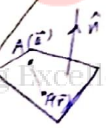
- If two lines are intersector, then $[(\vec{a} - \vec{c})\,\vec{b}\,\vec{d}] = 0$

Cartesian form:

$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ & $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$

Shortest dist $= \frac{\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$

Plane passing thru a point $\vec{a}$ & normal to vector $\hat{n}$:



$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

cartesian form: $r = (x, y, z)$, $a \equiv (x_1, y_1, z_1)$, $\hat{n} \rightarrow \overset{DR}{a, b, c}$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Plane passing thru 3 points $\vec{a}, \vec{b}$ & $\vec{c}$: $[\vec{AP}\ \vec{AB}\ \vec{AC}] = 0$

Plane passing thru points & parallel to vector $\vec{b}$ & $\vec{c}$:

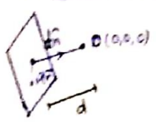
$[(\vec{r} - \vec{a})\,\vec{b}\,\vec{c}] = 0$

Plane passing thru point & a line $\vec{r} = \vec{c} + \lambda\vec{b}$:

$[(\vec{r} - \vec{a})\ (\vec{a} - \vec{c})\ \vec{b}] = 0$

Plane:

Normal form:



$\vec{r} \cdot \hat{n} = d$

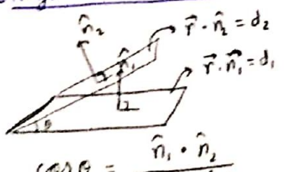
cartesian form:

$lx + my + nz = 0$

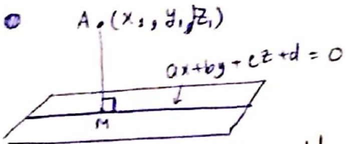
$r = x, y, z$

$\hat{n} \rightarrow l, m, n$

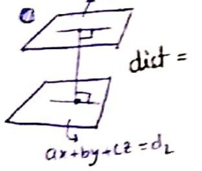
Angle b/w two planes:



$\cos\theta = \frac{\hat{n}_1 \cdot \hat{n}_2}{|\hat{n}_1||\hat{n}_2|}$

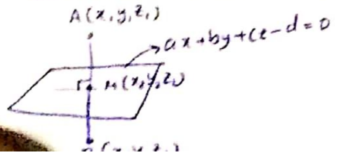


$AM = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$



dist $= \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

Foot of normal and image



$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = -\frac{[ax_1 + by_1 + cz_1 - d]}{a^2 + b^2 + c^2}$

$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = \frac{z_3 - z_1}{c} = -\frac{2|ax_1 + by_1 + cz_1 - d|}{a^2 + b^2 + c^2}$

- Convert the eq^n of line $(2x - y + z = 1$ & $x + y + 2z = 0)$ in Cartesian form.

⇒ let $z = t$

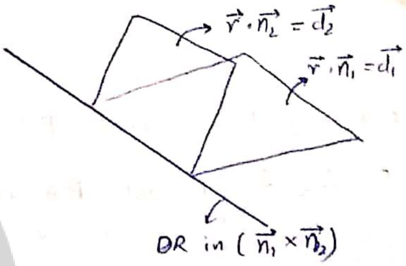
$$2x - y = 1 - t$$
$$x + y = -2t$$
$$3x = 1 - 3t$$
$$x = \frac{1-3t}{3}$$

$$y = -\frac{1-3t}{3} \longrightarrow t = \frac{1-3x}{3}$$

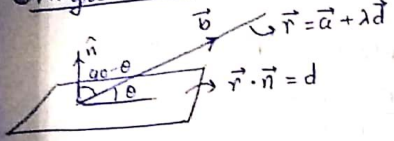
$$t = \frac{-1-3y}{3}$$

$$\therefore \frac{1-3x}{3} = \frac{-1-3y}{3} = z$$

$$\Rightarrow \frac{x - \frac{1}{3}}{-1} = \frac{y + \frac{1}{3}}{-1} = \frac{z-0}{1}$$



- Angle b/w plane & Line:

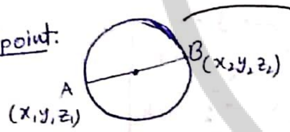


- Sphere: $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$\nearrow$ centre $(-u, -v, -w)$

$\searrow$ radius $= \sqrt{u^2 + v^2 + w^2 - d}$

→ Diametric point:



$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

PROBABILITY

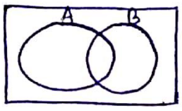
Probability of an event E, $P(E) = \frac{\text{Favourable outcomes}}{\text{Total no of outcomes}}$

$$\boxed{0 \leq P(E) \leq 1}$$

$P(E) = 0$ → Impossible event

$P(E) = 1$ → Certain event

$$\boxed{P(E) + P(\bar{E}) = 1}$$



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

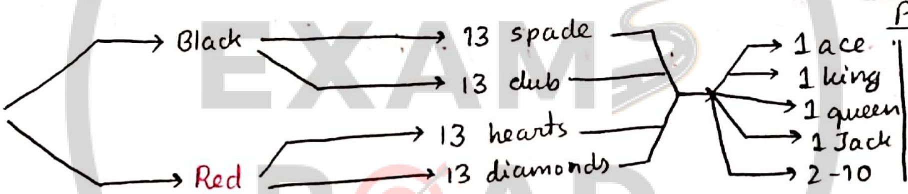
$A \cup B$ → A or B | $A \cap B$ → A and B

- **Mutually exclusive Events:** Only one of the events can occur at a time. $P(A \cap B) = 0$
- **Mutually independent Events:** Occurence of one event doesn't affect other events. $P(A \cap B) = P(A) \times P(B)$
- **Conditional Probability:** Prob of A given that B has already occured, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

  Prob of B ——— A ————, $P(B|A) = \frac{P(A \cap B)}{P(A)}$

4 suits → Heart, Spade, Club, Diamond

- **Playing Crads:** 52 cards



Honor card: Ace, king, Queen, Jack
(4×4=16)

Face card: King, Queen, Jack
(3×4=12)

Numbered cards: (2→10)
9×4 = 36

- **Baye's Theorem:**

$$P(E/E_2) = \frac{P(E) \times P(E_2/E)}{P(E_1)P(E_2/E_1) + P(E)P(E_2/E)}$$

E → Favoured event

E_1 → opposite of favoured event

E_2 → Already happened event.